TWO-GRID METHODS FOR SEMILINEAR INTERFACE PROBLEMS

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ABSTRACT. In this article we consider two-grid finite element methods for solving semilinear interface problems in d space dimensions, for $d = 2$ or $d = 3$. We first describe in some detail the target problem class with discontinuous diffusion coefficients, which includes problems containing sub-critical, critical, and supercritical nonlinearities. We then establish basic quasi-optimal *a priori* error estimate for Galerkin approximations. In the critical and subcritical cases, we follow our recent approach to controling the nonlinearity using only pointwise control of the continuous solution and a local Lipschitz property, rather than through pointwise control of the discrete solution; this eliminates the requirement that the discrete solution satisfy a discrete form of the maximum principle, hence eliminating the need for restrictive angle conditions in the underlying mesh. The supercritical case continues to require such mesh conditions in order to control the nonlinearity. We then design a two-grid algorithm consisting of a coarse grid solver for the original nonlinear problem, and a fine grid solver for a linearized problem. We analyze the quality of approximations generated by the algorithm, and show that the coarse grid may be taken to have much larger elements than the fine grid, and yet one can still obtain approximation quality that is asymptotically as good as solving the original nonlinear problem on the fine mesh. The algorithm we describe, and its analysis in this article, combines four sets of tools: the work of Xu and Zhou on two-grid algorithms for semilinear problems; the recent results for linear interface problems due to Li, Melenk, Wohlmuth, and Zou; and recent work on *a priori* estimates for semilinear problems.

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1. INTRODUCTION

In this article, we consider a two-grid finite element method for semilinear interface problems with discontinuous diffusion coefficients. One of the primary motivations of this work is to develop more efficient numerical methods for the nonlinear Poisson-Boltzmann equation, which has important applications in biochemistry and biophysics. However, the theory and techniques are applicable to a large class of semilinear interface problems, including problems with critical (and subcritical) nonlinearity arising in geometric analysis and mathematical general relativity.

In order to achieve our goal of exploiting two-grid-type discretizations, our first task is to more completely develop a basic quasi-optimal *a priori* error analysis for Galerkin approximations of semilinear interface problems. The main challenge comes from the loss of global regularity for interface problems (cf. [3, 17]). There has been much work on finite element approximation of the linear elliptic interface problem. For example, in [3] an equivalent minimization problem was introduced to handle the jump interface condition; this problem was then solved using finite element methods. Subsequently, finite element approximation of the elliptic interface problems was analyzed using penalty methods in [7], and optimal rates in the H^1 and L^2 norms were obtained by appropriately choosing the penalty parameter. Optimal *a priori* error estimates for linear interface problems in the energy norm (i.e., a weighted H^1 norm) is given in [25]. In [31], suboptimal error estimates of order $O(h|\log h|^{1/2})$ in the H^1 norm was obtained for 2D linear interface problems using standard finite element techniques. Similarly, in [11] it was shown that for C^2 interfaces in 2D convex polygonal domains Γ, the linear FEM approximation u_h has suboptimal standard error estimates of orders $O(h|\log h|^{1/2})$ and $O(h^2|\log h|^{1/2})$ in H^1 and L^2 norms respectively. By using isoparametric elements to fit the smooth interface, optimal error estimates were obtained in [27] for 2D interface problems. These results have been generalized to higher-order finite elements approximation in [21]. There are also other approaches for dealing with linear elliptic interface problem; for example, immersed interface finite element methods based on Cartesian grids (cf. [22]), mortar finite element (cf. [20]), and Lagrange multiplier methods using non-matching meshes (cf. [13]).

Less work has been done for nonlinear interface problems. For smooth coefficients under quite strong (global) regularity assumptions, quasi-optimal error estimates were obtained by [32, 34]. Due to the loss of global regularity for interface problems (cf. [3, 17], see also [23, 24] for regularity of linear interface problems), these analysis techniques are not applicable here. Recently, Sinha and Deka [28] studied linear finite element approximation of semilinear elliptic interface problems in two dimensional convex polygonal domains. Under assumptions that the mesh resolves the interface, and that the nonlinear function $b(\xi)$ satisfies

$$|b'(\xi)| \leq C|\xi|, \qquad \text{and } |b''(\xi)| \leqslant C, \qquad \forall \xi \in \mathbb{R},$$

they showed optimal error estimates in the H^1 norm using the framework of [8], together with the results from [11].

In this paper, we use a more natural approach for semilinear interface problems which can be applied to a somewhat different but larger class of nonlinear problems than [28]. For ease of exposition, we assume that the triangulation resolves the interface, although this assumption may be weakened. The first step is to derive both continuous and discrete *a priori* L^∞ bounds for the continuous and discrete solutions in order to control the nonlinearity. While continuous L^∞ bounds are fairly standard under quite general assumptions on the nonlinearity (cf. Assumption 2.2), discrete *a priori* L^∞ bounds require additional mesh conditions on the triangulation (cf. Assumption 3.1). Based on *a priori* L^∞ control of the continuous and discrete solutions, we derive optimal *a priori* error estimates in both the H^1 and L^2 norms, with the help of a *Local Monotonicity* assumption on the nonlinearity. A similar approach was used in [9, 14] for the Poisson-Boltzmann equation. We note the mesh conditions play a key role in obtaining discrete maximum/minimum principles (cf. [18, 15, 16, 30]). However, when the nonlinearity satisfies subcritical or critical growth conditions, and has some type of monotonicity, we have been able to derive quasi-optimal *a priori* error estimates directly, without using discrete maximum principles, and hence without the need for any mesh angle condition assumptions [4].

Finite element approximation of semilinear interface problems results in the need to solve system of nonlinear algebraic equations, and the number of unknowns in these systems can be extraordinary large in the case of three or more spatial dimensions. The most robust and efficient approach for solving these types of nonlinear algebraic systems has been repeatedly shown to be some variation of damped inexact Newton iteration, which consists of an inner-outer iteration: an inner loop involving repeated linear solves, together with any outer loop involving a damped/inexact correction step. See for example [5, 6, 26], and also [1] for an application to nonlinear interface problems. The basic approach involves the solution of a linear system on the fine mesh at each Newton step. However, the two-grid algorithm proposed in [2, 32] takes another approach, which consists of a coarse grid solver for the original nonlinear problem, and a fine grid solver only involving for a linearized problem, which is effectively a one-step Newton update of the solution. The benefit of using this two-grid idea is that it significantly reduces the overall computation cost, since we only need to solve the nonlinear problem on a coarse grid, and we can solve the linear problem on the fine grid by using standard multigrid/multilevel methods for optimal complexity. The central question concerning the two-grid method is to how choose the coarse grid problem; in other words, how coarse can one make the nonlinear problem discretization, but still achieve nearly optimal approximation properties if solving the full nonlinear problem on the fine grid. Based on *a priori* H^1 and L^2 error estimates for semilinear interface problems, in this paper we show that the basic framework developed in [32, 33] allows us to establish, both theoretically and numerically, that one may choose a coarse grid with much larger mesh size than the fine grid in the case of semilinear interface problems.

The main contributions of this paper are as follows:

(1) We give a complete finite element error analysis for semilinear elliptic interface problems, under weak assumptions on the nonlinearity; this includes establishing quasi-optimal *a priori* energy, L^2 and L^4 error estimates of the finite element approximation.
(2) We also provide a practical approach to efficiently solve the resulting nonlinear algebraic problem by two-grid algorithms, reducing the solution of the original nonlinear system of equations on the fine grid to the solution of a nonlinear problem on a coarse grid having much fewer degrees of freedom, together with the solution of a linear problem on the fine mesh. We note that the resulting linear interface problem can be efficiently solved by PCG algorithms using multilevel or domain decomposition preconditioners (cf. [35, 36]).

The remainder of the article is organized as follows. In Section 2, we introduce the basic notation and the model problem. We also establish continuous L^∞ bounds for the solution under very weak assumptions on the data and the nonlinearity. In Section 3, we establish quasi-optimal error estimates for the finite element approximation, by first deriving discrete *a priori* L^∞ bounds. In Section 4, we describe the two-grid algorithm, and give an analysis of the approximation properties of the algorithm. In Section 5, we give some numerical experiments to support our theoretical conclusions.

2. Semilinear Interface Problems

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain with $d \geqslant 2$, with an internal interface Γ dividing it into two open disjoint subdomains Ω_1 and Ω_2, so that $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. For ease of exposition, we assume Ω_1 and Ω_2 are two non-overlapping polyhedral/polygonal subdomains. We then focus on the following semilinear elliptic equation:

$$-\nabla \cdot (D\nabla u) + b(x,u) = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{2.1}$$

with the jump conditions on Γ:

$$[u] = 0, \text{ and } \left[D\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{on } \Gamma, \tag{2.2}$$

where $[u] := u_1|_\Gamma - u_2|_\Gamma$ and $\left[D\frac{\partial u}{\partial \mathbf{n}}\right] := D_1\frac{\partial u_1}{\partial \mathbf{n}_1} + D_2\frac{\partial u_2}{\partial \mathbf{n}_2}$ with $\mathbf{n}_i$ representing the unit outer normal on Ω_i. Here $u_i \ (i = 1,2)$ stands for the restriction of u on Ω_i. We assume that the coefficient $D = D(x) : \Omega \to \mathbb{R}^{d\times d}$ is symmetric and piecewise constant on each subdomain, i.e., $D = D_i$ in Ω_i, and $D \in L^\infty(\Omega)$ satisfies

$$m|\xi|^2 \leqslant \xi^T D(x)\xi \leqslant M|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \Omega, \tag{2.3}$$

for constants $m, M > 0$.

In working with the solution and approximation theory for (2.1)-(2.2), we will employ standard notation for the function spaces, norms, and other objects that will be relevant. For example, given any subset $G \subset \mathbb{R}^d$, we denote as $L^p(G)$ the Lebesgue spaces for $1 \leqslant p \leqslant \infty$, with norm $\|\cdot\|_{0,p,G}$. We denote the Sobolev norms as $\|v\|_{s,p,G} = \|v\|_{W^{s,p}(G)}$ for the spaces $W^{s,p}(G)$, with $W^{s,2}(G) = H^s(G)$ when $p = 2$. For any two functions $v \in L^p(G)$ and $w \in L^q(G)$ with $p, q \geqslant 1$ and $1/p + 1/q = 1$, we denote the pairing $(v,w)_G := \int_G vw dx$. For simplicity, when $G = \Omega$, we omit it in the norms/pairings. We will also denote as $H^s(\Omega_1 \cup \Omega_2)$ the space of functions u such that $u|_{\Omega_i} \in H^s(\Omega_i)$ for $i = 1,2$ and $s > 1$, endowed with the norm

$$\|u\|^2_{H^s(\Omega_1\cup\Omega_2)} := \|u\|^2_{H^s(\Omega_1)} + \|u\|^2_{H^s(\Omega_2)}.$$

We will use the notation $x_1 \lesssim y_1$, and $x_2 \gtrsim y_2$, whenever there exist constants C_1, C_2 independent of the mesh size h and the coefficient D or other parameters that x_1, x_2, y_1 and y_2 may depend on, and such that $x_1 \leqslant C_1 y_1$ and $x_2 \geqslant C_2 y_2$. We also denote $x \simeq y$ as $C_1 x \leqslant y \leqslant C_2 x$. Without confusion, we will also write $b(\xi) := b(x,\xi)$ and $b'(\xi) := \partial b(x,\xi)/\partial\xi$ for simplicity.

Remark 2.1. *Note that more general interface conditions* $\left[D\frac{\partial u^l}{\partial n_\Gamma}\right]_\Gamma = g_\Gamma$ *for some given function* $g_\Gamma \in L^\infty(\Gamma)$ *and non-homogeneous Dirichlet data* $u|_{\partial\Omega} = g$ *can be easily treated using our results here, due to the observation that one may split the equation into two sub-problems. The first sub-problem is a linear elliptic interface problem, and the second sub-problem is a nonlinear elliptic problem* (2.1) *with homogeneous Dirichlet boundary condition. More precisely, let* $u = u^l + u^n$, *where* $u^l \in H^1_g(\Omega)$ *satisfies the* *linear elliptic interface problem:*

$$\begin{cases} -\nabla\cdot(D\nabla u^l) & = 0 \text{ in } \Omega \\ u^l|_{\partial\Omega} = g, \text{ and } \left[D\frac{\partial u^l}{\partial n_\Gamma}\right]_\Gamma & = g_\Gamma; \end{cases} \tag{2.4}$$

while the nonlinear part u^n is the solution to the (homogeneous) semilinear equation

$$-\nabla\cdot(D\nabla u^n) + b(u^n + u^l) = 0 \text{ in } \Omega,$$

with the interface condition (2.2)*. On the other hand, the treatment for the linear interface problem* (2.4) *is standard; cf.* [11, 21]*. Therefore, without loss of generality we focus on* (2.1) *with homogeneous interface conditions* (2.2)*.*

The weak form of (2.1) reads: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) + (b(u), v) = 0, \qquad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where $a(u,v) := \int_\Omega D\nabla u\cdot\nabla v dx$. By the assumption (2.3) on the coefficient D, the bilinear form $a(u,v)$ in (2.5) is coercive and continuous, namely,

$$m\|\nabla u\|_{0,2}^2 \leqslant a(u,u), \qquad a(u,v) \leqslant M\|\nabla u\|_{0,2}\|\nabla v\|_{0,2}, \qquad \forall u,v\in H_0^1(\Omega), \tag{2.6}$$

where $0 < m \leqslant M < \infty$ are constants depending only on the maximal and minimal eigenvalues on D and the domain Ω. The properties (2.6) imply the semi-norm on $H_0^1(\Omega)$ is equivalent to the energy norm $|||\cdot||| : H_0^1(\Omega) \to \mathbb{R}$,

$$|||u|||^2 = a(u,u), \qquad m\|\nabla u\|_{0,2}^2 \leqslant |||u|||^2 \leqslant M\|\nabla u\|_{0,2}^2. \tag{2.7}$$

A priori L^∞ bounds for any solution to the continuous problem play a crucial role in controlling the nonlinearity. The following weak assumption on the nonlinearity allows for a large class of nonlinear problems containing both monotone and non-monotone nonlinearity:

Assumption 2.2. $b : \Omega\times\mathbb{R} \to \mathbb{R}$ *is a Carathéodory function, which satisfies the barrier-sign conditions in its second argument: there exist constants $\alpha, \beta \in \mathbb{R}$, with $\alpha \leqslant \beta$, such that*

$$\begin{aligned} b(x,\xi) \geqslant 0, \quad \forall \xi \geqslant \beta, \quad \text{a.e. in } \Omega \\ b(x,\xi) \leqslant 0, \quad \forall \xi \leqslant \alpha, \quad \text{a.e. in } \Omega. \end{aligned}$$

We have the following theorem based on the Assumptions 2.2:

Theorem 2.3 (*A Priori* L^∞ Bounds)**.** *Let the Assumption 2.2 hold. Let $u \in H_g^1(\Omega)$ be any weak solution to* (2.5)*. Then*

$$\underline{u} \leqslant u \leqslant \overline{u}, \quad \text{a.e. in } \Omega, \tag{2.8}$$

for the constants $\overline{u}$ and $\underline{u}$ defined by

$$\overline{u} := \max\left\{\beta, \sup_{x\in\partial\Omega} g(x)\right\}, \qquad \underline{u} := \min\left\{\alpha, \inf_{x\in\partial\Omega} g(x)\right\}, \tag{2.9}$$

where $\alpha \leqslant \beta$ are the constants in Assumption 2.2.

Proof. To prove the upper bound, let us introduce

$$\phi = (u-\overline{u})^+ = \max\{u-\overline{u}, 0\}.$$

By the definition of $\overline{u}$, it follows (cf. [29, Theorem 10.3.8]) that $\phi \in H_0^1(\Omega)$ and $\phi \geqslant 0$ a.e. in Ω. Taking $v = \phi$ in (2.5), we have

$$a(u,\phi) = a(u-\overline{u},\phi) = a(\phi,\phi) \geqslant m\|\nabla\phi\|_{0,2}^2.$$

This implies that

$$m\|\nabla\phi\|_{0,2}^2 \leqslant a(u,\phi) = (-b(u),\phi) \leqslant 0,$$

since $-b(u) \leqslant 0$ a.e. in the support of ϕ. Hence, $\|\nabla\phi\|_{0,2} \equiv 0$ which yields $\phi = 0$. Therefore, the upper bound of (2.8) holds.

Similarly, we introduce

$$\psi = (u - \underline{u})^- = \min\{u - \underline{u}, 0\}.$$

It is obvious that $\psi \in H_0^1(\Omega)$ can be used as a test function in (2.5). Moreover, $\psi \leqslant 0$ a.e. in Ω, and Assumption 2.2 implies $-b(u) \geqslant 0$ on the support of ψ. Therefore,

$$m\|\nabla\psi\|_{0,2}^2 \leqslant a(u,\psi) = (-b(u),\psi) \leqslant 0,$$

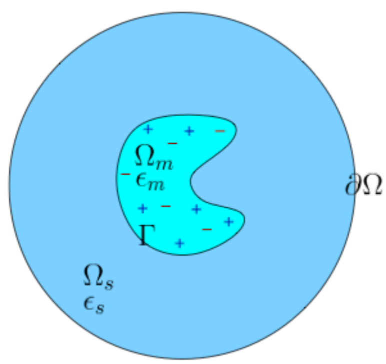
which implies $\psi \equiv 0$ as before. This proves the lower bound of (2.8). □

To conclude this section, we give the nonlinear Poisson-Boltzmann equation as an example, which is one of our main motivation for this work. This equation has been widely used in biochemistry, biophysics and in semiconductor modeling for describing the electrostatic interactions of charged bodies in dielectric media.

Example 2.4. *The regularized Poisson-Boltzmann equation reads:*

$$\begin{cases} -\nabla\cdot(\varepsilon\nabla u) + \kappa^2\sinh(u) &= 0, \quad \text{in } \Omega \\ [u]_\Gamma = 0 \text{ and } \left[\varepsilon\frac{\partial u}{\partial \mathbf{n}}\right]_\Gamma &= g_\Gamma, \quad \text{on } \Gamma \\ u|_{\partial\Omega} &= g, \quad \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

where $g_\Gamma \in L^\infty(\Gamma)$ is a function defined on Γ arising from regularization of pointwise charges in the molecular region (see [9, 14] for detailed derivations). Here the diffusion coefficient ε is piecewise positive constant $\varepsilon|_{\Omega_1} = \varepsilon_1$ and $\varepsilon|_{\Omega_2} = \varepsilon_2$, where Ω_1 is the molecular region, and Ω_2 is the solution region(see Figure 2.4 for example). The modified



Debye-Hückel parameter κ^2 is also piecewise constant $\kappa^2(x)|_{\Omega_1} = 0$ and $\kappa^2(x)|_{\Omega_2} > 0$. The Dirichlet condition $u|_{\partial\Omega} = g$ are imposed on the boundary $\partial\Omega$. We note that equation (2.10) can be reduced to (2.1) by splitting it into linear and nonlinear components as described in Remark 2.1, see [14] for more details. Obviously, the Assumption (A1) is satisfied for (2.10).

3. Finite Element Error Estimates

We now discuss some error estimates on the finite element discretization of (2.5) which will play a key role in the two-grid analysis. Given a quasi-uniform triangulation $\mathcal{T}_h$ of Ω, we denote by $V_h(g) \subset H_g^1(\Omega)$ the standard piecewise linear finite element space satisfying the Dirichlet boundary condition. For simplicity, we denote $V_h := V_h(0)$. For ease of exposition, we assume the triangulation $\mathcal{T}_h$ resolves the interface Γ. Then finite element approximation of the target problem (2.1) reads: Find $u_h \in V_h$ such that

$$a(u_h, v) + (b(u_h), v) = 0, \qquad \forall v \in V_h. \tag{3.1}$$

The following theorem shows that under appropriate mesh condition on $\mathcal{T}_h$, the discrete solution u_h of (3.1) satisfies *a priori* L^∞ bounds (as does the continuous solution u due to Theorem 2.3). More precisely, assume the triangulation $\mathcal{T}_h$ satisfies

Assumption 3.1. *Let ϕ_i and ϕ_j are the basis functions corresponding to the vertices x_i and x_j, respectively. We assume that*

$$a(\phi_i, \phi_j) \leqslant 0, \qquad \forall i \neq j, \tag{3.2}$$

Under this assumption, we can obtain the following *a priori* L^∞ bound of the discrete solution u_h.

Theorem 3.2. *Let b satisfy the Assumption 2.2, and $\mathcal{T}_h$ satisfy the Assumption 3.1. Then the discrete solution $u_h \in V_h(g)$ to* (3.1) *satisfies*

$$\underline{u} \leqslant u_h \leqslant \overline{u}, \quad a.e. \quad in\, \Omega, \tag{3.3}$$

where $\underline{u}$ and $\overline{u}$ are defined in (2.9).

The idea of the proof of (3.3) is the same as in Theorem 2.3. However, in the discrete setting, for a given $r \in \mathbb{R}$ the truncated functions $(u_h - r)^\pm$ are usually not in V_h. Thus, they can not be used as test functions in (3.1). Instead, one can employ the nodal interpolation of these functions as test functions. In particular, given any constant r, we denote

$$[u_h - r]^\pm := \sum_{i=1}^{N} (u_h(x_i) - r)^\pm \phi_i,$$

where N is the total number of degree of freedoms, and ϕ_i ($i = 1, \cdots, N$) is the nodal basis function at the vertex x_i. While this does produce proper test functions, it unfortunately introduces mesh conditions such as Assumption 3.1 into the analysis.

Proof of Theorem 3.2. To prove the upper bound of (3.3), define a test function $\phi_h^+(x) := [u_h(x) - \overline{u}]^+$. It is obvious that $\phi_h^+ \in V_h$ and $\mathrm{supp}(\phi_h^+)$ is the union of of the macro elements for the vertices such that $u_h(x_i) > \overline{u}$. Therefore, it satisfies

$$a(u_h, \phi_h^+) + (b(u_h), \phi_h^+) = 0. \tag{3.4}$$

For the diffusion term in (3.4), we notice that

$$\begin{aligned} a(u_h, \phi_h^+) &= a(\phi_h^+, \phi_h^+) + a([u_h - \overline{u}]^-, \phi_h^+) \\ &= a(\phi_h^+, \phi_h^+) + \sum_{i \neq j} (u_h(x_i) - \overline{u})^- (u_h(x_j) - \overline{u})^+ a(\phi_i, \phi_j) \\ &\geqslant a(\phi_h^+, \phi_h^+) \geqslant m \|\nabla \phi_h^+\|_{0,2}^2, \end{aligned}$$

where we used the Assumption 3.1 in the third step, and used (2.3) in the last step. Thus we obtain that

$$m \|\nabla \phi_h^+\|_{0,2}^2 \leqslant a(u_h, \phi_h^+) = (-b(u_h), \phi_h^+) \leqslant 0.$$

This implies that $\phi_h^+ \equiv 0$, and hence $u_h \leqslant \overline{u}$ a.e. in Ω. The proof of the lower bound is similar, and so we omit the detail here. $\square$

Remark 3.3. *Note that Assumption 3.1 requires certain angle condition on the triangulation $\mathcal{T}_h$. This condition is crucial in proving the discrete maximal/minimal principle (cf. [12, 18, 15, 30]). However, in case that b satisfies critical/subcritical growth condition, namely, there exists some constant $K > 0$ such that*

$$|b^{(n)}(\xi)| \leq K, \quad \forall \xi \in \mathbb{R} \tag{3.5}$$

where n is an integer satisfying $n < \infty$ when $d = 2$ and $n \leqslant (d+2)/(d-2)$ when $d \geqslant 3$, we are able to show the quasi-optimal error estimate directly, without using Assumption 3.1; see [4] for more detail.

A priori L^∞ bounds (Theorem 2.3 and Theorem 3.2) play crucial roles in controlling the nonlinearity, ensuring that the nonlinearity b has a certain "local Lipschitz" property. This property in turn is used to establish quasi-optimal error estimates for the finite element approximations. For this purpose, let us make the following additional assumption on b :

Assumption 3.4. *b is locally monotone, namely,*

$$b'(\xi) \geqslant 0, \qquad \forall \xi \in [\underline{u}, \overline{u}], \tag{3.6}$$

where $\underline{u}, \overline{u}$ are the barriers defined in (2.9).

Without loss of generality, in the remainder of the paper, we let the Dirichlet data $g = 0$. With the help of the *a priori* L^∞ bounds of u and u_h in Theorem 2.3 and Theorem 3.2 respectively, we are able to establish the following quasi-optimal error estimate.

Theorem 3.5. *Let b satisfies the Assumptions 2.2 and 3.4, and $\mathcal{T}_h$ satisfy the Assumption 3.1. Let $u \in H_0^1(\Omega) \cap H^s(\Omega_1 \cup \Omega_2)$ for some $s > 1$ and $u_h \in V_h$ be the solution to (2.5) and the discrete solution to (3.1), respectively. Then the following quasi-optimal error estimate holds:*

$$|||u - u_h||| \lesssim \inf_{v \in V_h} |||u - v||| \lesssim h^{s-1} \|u\|_{H^s(\Omega_1 \cup \Omega_2)}. \tag{3.7}$$

Proof. By Assumptions 2.2 and 3.1, Theorem 2.3 and 3.2 give *a priori* L^∞ bounds on u and u_h:

$$\underline{u} \leqslant u, u_h \leqslant \overline{u}$$

for the constants $\underline{u} \leqslant \overline{u}$ defined in (2.9). This implies that

$$b(u) - b(u_h) = b'(\xi)(u - u_h) \leqslant C_L |u - u_h|, \tag{3.8}$$

where $C_L = \sup_{\xi \in [\underline{u}, \overline{u}]} \|b'(\xi)\|_{0,\infty}$ is a constant depending only on b, $\underline{u}$ and $\overline{u}$. Subtracting equation (3.1) from (2.5), we have

$$a(u - u_h, v) + (b(u) - b(u_h), v) = 0, \qquad \forall v \in V_h.$$

By using this identity, we obtain

$$\begin{aligned}
|||u - u_h|||^2 &= a(u - u_h, u - u_h) \\
&= a(u - u_h, u - v) + a(u - u_h, v - u_h), \quad \forall v \in V_h \\
&= a(u - u_h, u - v) + (b(u) - b(u_h), u_h - v) \\
&= a(u - u_h, u - v) + (b(u) - b(u_h), u - v) - (b'(\xi)(u - u_h), u - u_h) \\
&\leqslant |||u - u_h||| \, |||u - v||| + C_L \|u - u_h\|_{0,2} \|u - v\|_{0,2},
\end{aligned}$$

where in the last inequality, we used Cauchy-Schwarz inequality, the Lipschitz property (3.8) of b and the Local Monotonicity (3.6) from Assumption 3.4. Then by Poincaré inequality we have

$$|||u-u_h|||^2 \leqslant |||u-u_h||| \, |||u-v||| + C_L C_P^2 \|\nabla(u-u_h)\|_{0,2} \|\nabla(u-v)\|_{0,2}$$

where C_P is the Poincaré constant. Thus we obtain

$$|||u-u_h||| \lesssim |||u-v|||.$$

Therefore, we have proved the first inequality in (3.7), since $v \in V_h$ is arbitrary. The second inequality in (3.7) follows by standard interpolation error estimates; cf. [21, Theorem 3.5]. □

To conclude this section, let us try to derive L^2 error estimates: $\|u-u_h\|_{0,2}$, by using duality arguments. To begin, introduce the following linear adjoint problem: Find $w \in H_0^1(\Omega)$ such that

$$a(v,w) + (b'(u)v, w) = (u-u_h, v), \qquad v \in H_0^1(\Omega). \tag{3.9}$$

We assume that the linear interface problem (3.9) has the regularity

$$\|w\|_{H^\tau(\Omega_1\cup\Omega_2)} \leqslant C\|u-u_h\|_{0,2} \tag{3.10}$$

for some $\tau > 1$. The regularity assumption (3.10) is also called "τ-regularity" in [21, Assumption 4.3], which is quite natural for linear interface problems. Along with (3.9), let us also introduce the finite element approximation $w_h \in V_h$ which satisfies:

$$a(v_h, w_h) + (b'(u)v_h, w_h) = (u-u_h, v_h), \qquad v_h \in V_h. \tag{3.11}$$

Then by standard finite element approximation theory for the linear interface problem (3.9) (cf. [11, 21]), we have the following error estimate:

$$|||w-w_h||| \lesssim h^{\tau-1}\|w\|_{H^\tau(\Omega_1\cup\Omega_2)} \lesssim h^{\tau-1}\|u-u_h\|_{0,2}, \tag{3.12}$$

where in the second inequality we have used the regularity assumption (3.10). We then have the following L^2 error estimate for u_h:

Theorem 3.6 (L^2 Error Estimate)**.** *Let b satisfy Assumptions 2.2 and 3.4, and let $\mathcal{T}_h$ satisfy Assumption 3.1. Let $u \in H_0^1(\Omega) \cap H^s(\Omega_1 \cup \Omega_2)$ with $s > 1$ be the solution to (2.5), and let u_h be the solution to (3.1). Suppose that the dual problem (3.9) satisfies the τ-regularity (3.10) for some $\tau > 1$. Then*

$$\|u-u_h\|_{0,2} \leqslant C(h^{\tau-1} + h^{s-1})\|\nabla(u-u_h)\|_{0,2}, \tag{3.13}$$

where C is independent of h.

Proof. Without loss of generality we assume $b''(\chi) \neq 0$. By taking $v = u-u_h$ in (3.9) we obtain that

$$\begin{aligned}\|u-u_h\|_{0,2}^2 &= a(u-u_h, w) + (b'(u)(u-u_h), w)\\ &= a(u-u_h, w-w_h) + (b'(u)(u-u_h), w-w_h)\\ &\quad + a(u-u_h, w_h) + (b'(u)(u-u_h), w_h).\end{aligned} \tag{3.14}$$

To bound the second term in (3.14), we use the L^∞ bound of u (cf. Theorem 2.3) to obtain

$$\begin{aligned}(b'(u)(u-u_h), w-w_h) &\leqslant \|b'(u)\|_{0,\infty}\|u-u_h\|_{0,2}\|w-w_h\|_{0,2}\\ &\lesssim \|\nabla(u-u_h)\|_{0,2}\|\nabla(w-w_h)\|_{0,2},\end{aligned}$$

where we have used Poincaré inequality for $(u-u_h)$ and $(w-w_h)$ in the last step. To deal with the last two terms in (3.14), notice that $u_h \in V_h$ is the solution to the discrete semilinear problem (3.1), we have

$$a(u-u_h, w_h) = -(b(u)-b(u_h), w_h).$$

Thus by Taylor expansion, we have

$$\begin{aligned} a(u-u_h, w_h) + (b'(u)(u-u_h), w_h) &= -(b(u)-b(u_h) - (b'(u)(u-u_h), w_h) \\ &= \frac{1}{2}(b''(\chi)(u-u_h)^2, w_h), \end{aligned}$$

where χ satisfies that $\underline{u} \leqslant \chi(x) \leqslant \overline{u}$ a.e. in Ω due to the *a priori* L^∞ bounds of u and u_h by Theorem 2.3 and Theorem 3.2. Therefore, by Hölder inequality the last two terms in (3.14) can be bounded as:

$$\begin{aligned} a(u-u_h, w_h) + (b'(u)(u-u_h), w_h) &\leqslant \frac{1}{2}\|b''(\chi)\|_{0,\infty}\|u-u_h\|^2_{0,p^*}\|w_h\|_{0,q*} \\ &\lesssim \|\nabla(u-u_h)\|^2_{0,2}\|w_h\|_{0,q^*}, \end{aligned}$$

where we choose $p^* = 6$ for $d = 3$ and $p^* > 4$ when $d = 2$, and $\frac{2}{p^*} + \frac{1}{q^*} = 1$. In the last inequality, we have used the Sobolev embedding $\|u-u_h\|_{0,p^*} \lesssim \|\nabla(u-u_h)\|_{0,2}$. Therefore, we obtain

$$\|u-u_h\|^2_{0,2} \lesssim (\|\nabla(w-w_h)\|_{0,2} + \|\nabla(u-u_h)\|_{0,2}\|w_h\|_{0,q^*})\|\nabla(u-u_h)\|_{0,2}. \quad (3.15)$$

Now in (3.15), let $w_h \in V_h$ be the solution to (3.11). Then the estimate for the quantity $\|\nabla(w-w_h)\|_{0,2}$ is readily available from (3.12). It then remains to estimate the term $\|\nabla(u-u_h)\|_{0,2}\|w_h\|_{0,q^*}$. To estimate $\|w_h\|_{0,q^*}$, notice that by the choice of p^*, $1 < q^* < 2$. Then by Poincaré inequality and coercivity of $a(\cdot,\cdot)$

$$\|w_h\|_{0,q^*} \lesssim \|w_h\|_{0,2} \lesssim \|\nabla w_h\|_{0,2} \lesssim |||w_h|||.$$

By Assumption 3.4 on b, we have

$$\begin{aligned} |||w_h|||^2 &= a(w_h, w_h) \leqslant a(w_h, w_h) + (b'(u)w_h, w_h) \\ &= (u-u_h, w_h) \leqslant \|u-u_h\|_{0,2}\|w_h\|_{0,2} \\ &\lesssim \|u-u_h\|_{0,2}|||w_h|||. \end{aligned}$$

Thus, this inequality implies that

$$\|w_h\|_{0,q^*} \lesssim |||w_h||| \lesssim \|u-u_h\|_{0,2}.$$

Therefore, we obtain

$$\|\nabla(u-u_h)\|_{0,2}\|w_h\|_{0,q^*} \lesssim h^{s-1}\|u\|_{H^s(\Omega_1\cup\Omega_2)}\|u-u_h\|_{0,2}. \quad (3.16)$$

Combining inequalities (3.15), (3.12) and (3.16), the inequality (3.13) then follows. This concludes the proof. □

Remark 3.7. *Theorem 3.6 implies, in particular, that*

$$\|u-u_h\|_{0,2} \lesssim h^t\|\nabla(u-u_h)\|_{0,2}, \quad (3.17)$$

where $t = \min\{s,\tau\} - 1$. *Combining with the quasi-optimal error estimate* (3.7), *we obtain the following* L^2 *error estimate:*

$$\|u-u_h\|_{0,2} \lesssim h^{t+(s-1)}\|u\|_{H^s(\Omega_1\cup\Omega_2)}. \quad (3.18)$$

Remark 3.8. *Similar to Remark 3.3, if* $b(\cdot)$ *satisfies the critical/subcritical growth condition* (3.5), *then the conclusion of Theorem 3.6 holds without the Assumption 3.1. We refer to* [4] *for the details.*

4. Two-grid algorithms

We now consider a two-grid algorithm (cf. [32]) to solve the finite element discretization (3.1) numerically. Let $\mathcal{T}_H$ be a quasi-uniform triangulation with mesh size H, and $\mathcal{T}_h$ with mesh size $h < H$ is a uniform refinement of $\mathcal{T}_H$. We assume the triangulations satisfy Assumption (3.1). The algorithm consists of an *exact* coarse solver on a coarse grid $\mathcal{T}_H$, and a Newton update on the fine grid $\mathcal{T}_h$. In what follows, we will denote u_h, u_H as the *exact* finite element solutions to (3.1) on the grids $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. For simplicity, let us denote

$$\langle F(u), \chi\rangle := a(u,\chi) + (b(u),\chi),$$

and its linearization

$$\langle F'(u)v, \chi\rangle := a(v,\chi) + (b'(u)v,\chi).$$

The two grid algorithm considered in this paper is as follows: The Algorithm 1 solves

Algorithm 1: $u^h = \mathsf{TwoGrid}\,(\mathcal{T}_H, \mathcal{T}_h)$

1 Find $u_H \in V_H$ such that

$$\langle F(u_H), v_H\rangle = 0, \quad \forall v_H \in V_H;$$

2 Find $u^h \in V_h$ such that

$$\langle F'(u_H)u^h, v_h\rangle = \langle F'(u_H)u_H, v_h\rangle - \langle F(u_H), v_h\rangle \quad \forall v_h \in V_h;$$

the original nonlinear problem on the coarse grid $\mathcal{T}_H$, and then performs one Newton iteration on the fine grid.

Fix any $\chi_h \in V_h$, let $\eta(t) := \langle F(u_H + t(u_h - u_H)), \chi_h\rangle$. Then by Taylor expansion, we have

$$\begin{aligned} 0 &= \langle F(u_h), \chi_h\rangle = \eta(1) = \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1-t)dt \\ &= \langle F(u_H), \chi_h\rangle + \langle F'(u_H)(u_h - u_H), \chi_h\rangle + \int_0^1 \eta''(t)(1-t)dt. \end{aligned}$$

Notice that by direct calculation, the remainder term, denoted by $R(u_H, u_h, \chi_h)$, has the following form:

$$\begin{aligned} R(u_H, u_h, \chi_h) &:= \int_0^1 \eta''(t)(1-t)dt \\ &= \int_0^1 (b''(u_H + t(u_h - u_H))(u_h - u_H)^2, \chi_h)dt. \end{aligned}$$

By Theorem 3.2, we have $u_h, u_H \in [\underline{u}, \overline{u}]$. Therefore, we have the following estimate on $R(u_H, u_h, \chi_h)$:

$$|R(u_H, u_h, \chi_h)| \le C\|u_H - u_h\|_{0,2p}^2 \|\chi_h\|_{0,q}, \tag{4.1}$$

for any $p, q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 4.1. *Let b satisfy the Assumption 2.2 and 3.4, and $\mathcal{T}_h, \mathcal{T}_H$ satisfy the Assumption 3.1. Let $u_H \in V_H$ and $u_h \in V_h$ be the exact solutions to (3.1) on $\mathcal{T}_H$ and $\mathcal{T}_h$ respectively, and $u^h \in V_h$ be the approximated solution obtained by Algorithm 1. Then, we have the following estimate*

$$|\!|\!| u_h - u^h |\!|\!| \lesssim \|u_h - u_H\|_{0,4}^2. \tag{4.2}$$

Proof. From Algorithm 1, we have

$$\begin{aligned}\langle F'(u_H)(u_h - u^h), \chi_h\rangle &= \langle F'(u_H)(u_h - u_H), \chi_h\rangle + \langle F(u_H), \chi_h\rangle \\ &= -R(u_H, u_h, \chi_h), \quad \forall \chi_h \in V_h.\end{aligned}$$

Then by taking $\chi_h = u_h - u^h \in V_h$ in the above equality, we obtain that

$$\begin{aligned}|||u_h - u^h|||^2 &= a(u_h - u^h, u_h - u^h) \\ &= \langle F'(u_H)\chi_h, \chi_h\rangle - (b'(u_H)\chi_h, \chi_h) \\ &\leqslant -R(u_H, u_h, u_h - u^h) \\ &\lesssim \|u_h - u_H\|_{0,2p}^2 \|u_h - u^h\|_{0,q},\end{aligned}$$

where in the third step we used the Assumption 3.4, and in the last step we used (4.1). In particular, if we pick $p = q = 2$, the conclusion then follows by Poincaré inequality on $\|u_h - u^h\|_{0,2}$. □

Lemma 4.1 suggests that we will need the L^4 error estimates $\|u - u_h\|_{0,4}$. For this purpose, we make use of the following Ladyzhenskaya's inequalities:

Lemma 4.2 ([19, Lemma 1-2])**.** *For any $v \in H_0^1(\Omega)$, it holds*

$$\|v\|_{0,4} \leqslant \sqrt[4]{2}\|v\|_{0,2}^{\frac{1}{2}}\|\nabla v\|_{0,2}^{\frac{1}{2}}, \qquad d = 2; \tag{4.3}$$

and

$$\|v\|_{0,4} \leqslant \sqrt{2}\|v\|_{0,2}^{\frac{1}{4}}\|\nabla v\|_{0,2}^{\frac{3}{4}}, \qquad d = 3. \tag{4.4}$$

Recall that we assume the solution to the original nonlinear problem (2.1) satisfies $u \in H_0^1(\Omega) \cap H^s(\Omega_1 \cup \Omega_2)$, and the dual linear problem (3.9) has the regularity $w \in H_0^1(\Omega) \cap H^\tau(\Omega_1 \cup \Omega_2)$ for some $s, \tau > 1$. We let $t = \min\{s, \tau\} - 1$ as defined in (3.17). As a corollary of Lemma 4.2, we obtain the L^4 error estimate:

Corollary 4.3. *Let b satisfy the Assumptions 2.2 and 3.4, and $\mathcal{T}_h$ satisfy the Assumption 3.1. Let $u \in H_0^1(\Omega) \cap H^s(\Omega_1 \cup \Omega_2)$ with $s > 1$ be the solution to (2.5), and $u_h \in V_h$ be the solution to (3.1). Suppose that the dual problem (3.9) satisfies the τ-regularity (3.10) for some $\tau > 1$. Then the following error estimates hold:*

$$\|u - u_h\|_{0,4} \lesssim h^{\frac{t}{2}+(s-1)}\|u\|_{H^s(\Omega_1\cup\Omega_2)}, \qquad d = 2; \tag{4.5}$$

and

$$\|u - u_h\|_{0,4} \lesssim h^{\frac{t}{4}+(s-1)}\|u\|_{H^s(\Omega_1\cup\Omega_2)}, \qquad d = 3, \tag{4.6}$$

where $t = \min\{s, \tau\} - 1$.

Proof. The proof is simply a combination of Lemma 4.2 and the quasi-optimal error estimate (3.7) in Theorem 3.5 and the L^2 error estimate in (3.17). When $d = 2$, by (4.3) we have

$$\|u - u_h\|_{0,4} \leqslant \sqrt[4]{2}\|u - u_h\|_{0,2}^{\frac{1}{2}}\|\nabla(u - u_h)\|_{0,2}^{\frac{1}{2}} \lesssim h^{t/2}\|\nabla(u - u_h)\|_{0,2}.$$

Thus, when $d = 2$ we obtain

$$\|u - u_h\|_{0,4} \lesssim h^{\frac{t}{2}+(s-1)}\|u\|_{H^s(\Omega_1\cup\Omega_2)}$$

Similarly, when $d = 3$ by (4.4) and (3.17) we obtain

$$\|u - u_h\|_{0,4} \leq Ch^{\frac{t}{4}}\|\nabla(u - u_h)\|_{0,2}.$$

The conclusion then follows from the conclusion of Theorem 3.5. □

Finally, we obtain the following main result.

Theorem 4.4. *Let b satisfy the Assumptions 2.2 and 3.4, and $\mathcal{T}_h, \mathcal{T}_H$ satisfy the Assumption 3.1. Let $u \in H_0^1(\Omega) \cap H^s(\Omega_1 \cup \Omega_2)$ with $s > 1$ be the solution to (2.5), and u^h be the solution to Algorithm 1. Suppose that the dual problem (3.9) satisfies the τ-regularity (3.10) for some $\tau > 1$. We have the following estimates*

$$\|u - u^h\| \lesssim (h^{s-1} + H^{t+2(s-1)})\|u\|_{H^s(\Omega_1\cup\Omega_2)}, \qquad d = 2, \tag{4.7}$$

and

$$\|u - u^h\| \lesssim (h^{s-1} + H^{\frac{t}{2}+2(s-1)})\|u\|_{H^s(\Omega_1\cup\Omega_2)}, \qquad d = 3, \tag{4.8}$$

where $t = \min\{s, \tau\} - 1 > 0$.

Proof. By triangle inequality, we have

$$\begin{aligned}\|u - u^h\| &\le \|u - u_h\| + \|u_h - u^h\| \\ &\lesssim \|u - u_h\| + \|u_h - u_H\|_{0,4}^2 \\ &\lesssim \|u - u_h\| + \|u - u_h\|_{0,4}^2 + \|u - u_H\|_{0,4}^2. \end{aligned} \tag{4.9}$$

The first term in the right hand side of (4.9) has been estimated in Theorem 3.5. Thus the conclusions immediately follow by applying Corollary 4.3 to the last two terms in (4.9). □

Remark 4.5. *Based on Theorem 4.4, we may choose $H \leqslant h^{\frac{s-1}{t+2(s-1)}}$ in 2D and $H \le h^{\frac{s-1}{t/2+2(s-1)}}$ in 3D, but still achieve quasi-optimal error estimate. In particularly, if the linear dual problem (3.9) has the same or more regularity than the primal nonlinear problem (2.1), i.e., $\tau \geqslant s > 1$, then $t = s - 1$. So, in 2D case we may choose $H \leqslant h^{1/3}$ and $H \leqslant h^{2/5}$ in 3D.*

5. Numerical Experiments

In this section, we present some numerical experiments to justify the theories. Here we consider solving the following semilinear equation

$$-\nabla \cdot (D\nabla u) + u^{11} = 1000\delta_0, \quad u|_{\partial\Omega} = 0,$$

where the diffusion coefficient $D = 1000$ inside $[-1/2, 1/2]^2$ and 1 outside, and δ_0 is the delta function on origin. See Figure 1 for the solution of this equation. Figure 2 and 3

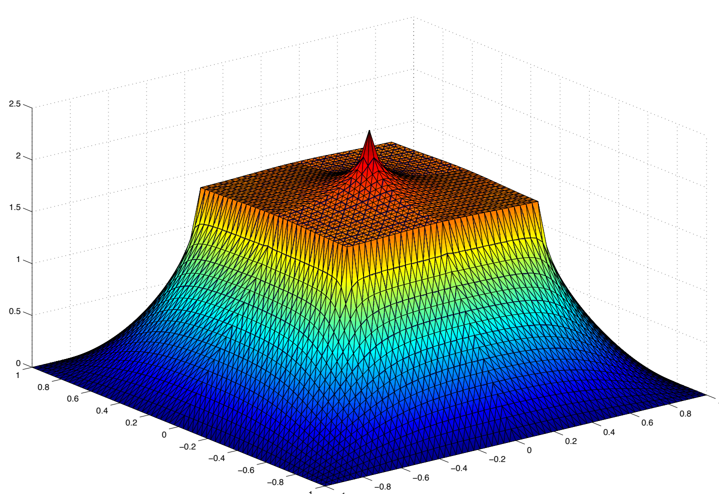


FIGURE 1. Solution to the 2D semilinear interface problem.

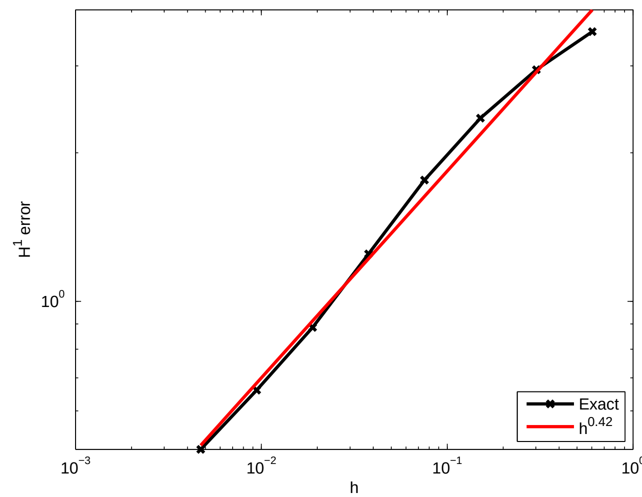


FIGURE 2. Errors in H^1-norm.

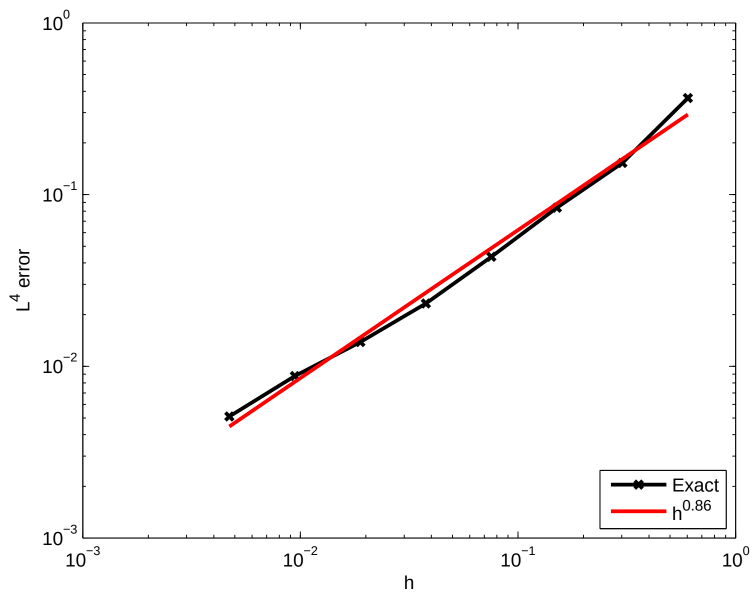


FIGURE 3. Errors in L^4 norm.

show the H^1 and L^4 errors, respectively. Figure 4 shows the comparison of the exact H^1 error with the error of the two-grid solution produced by the Algorithm 1. Here, the mesh size H of the coarse grid problem is chosen to be closest to the theoretical ones obtained from Theorem 4.4 if not exactly the same. As we can see from this figure, the two-grid solution is very close to the exact solution. Therefore, by the appropriately choice of the coarse problem, solving the nonlinear problem could be reduced to solving a linear problem on the fine mesh without loss of accuracy. Note that the linearized problem on the fine mesh could be solved efficiently by multilevel preconditioning techniques, even in the presence of large jump coefficients (cf. [35]). In this way, we reduced greatly the overall computational cost for solving the nonlinear PDEs.

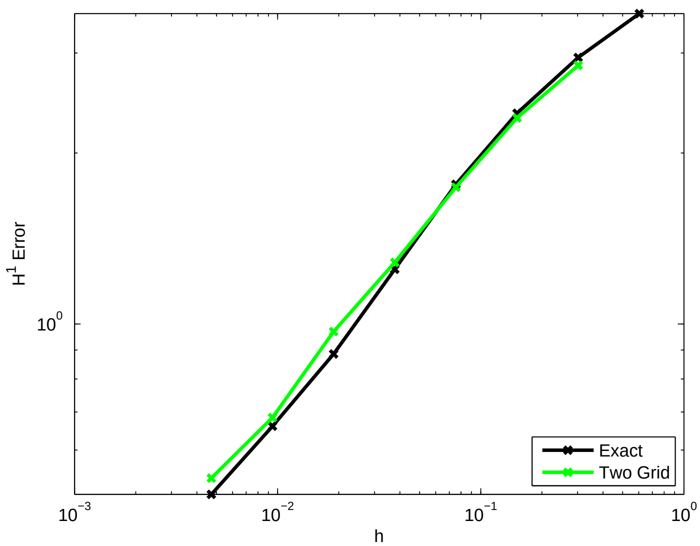


FIGURE 4. Two-grid Error.

6. CONCLUSION AND EXTENSION

In this article we considered a two-grid finite element method for solving semilinear interface problems in d space dimensions, for $d = 2$ or $d = 3$. We first described in some detail the target problem class with discontinuous diffusion coefficients, which included critical (and subcritical) nonlinearity examples, as well problems containing supercritical nonlinearity (such as the Poisson-Boltzmann equation and the semi-conductor device modeling equations). We then developed a basic quasi-optimal *a priori* error estimate for Galerkin approximations. In the critical and subcritical cases, we follow [4] and control the nonlinearity using only pointwise control of the continuous solution and a local Lipschitz property, rather than through pointwise control of the discrete solution; this eliminates the requirement that the discrete solution satisfy a discrete form of the maximum principle, hence eliminating the need for restrictive angle conditions in the underlying mesh. However, the supercritical case continues to require such conditions in order to control the nonlinearity. We then designed a two-grid algorithm consisting of a coarse grid solver for the original nonlinear problem, and a fine grid solver for a linearized problem. We analyzed the quality of approximations generated by the algorithm, and proved that the coarse grid may be taken to have much larger elements than the fine grid, and yet one can still obtain approximation quality that is asymptotically as good as solving the original nonlinear problem on the fine mesh. The included numerical experiments support our theoretical results.

The algorithm we described, and its analysis in this article, combined four sets of tools: the work of Xu and Zhou on two-grid algorithms for semilinear problems [32, 33, 34]; the recent results for linear interface problems due to Li, Melenk, Wohlmuth, and Zou [21]; recent work on the Poisson-Boltzmann equation [10, 14]; and recent results on *a priori* estimates for semilinear problems, including estimates without angle conditions in the case of sub- and super-critical nonlinearity [4]. Although the algorithm described in this paper is applicable to general coupled nonlinear elliptic systems, our reliance on tools developed for scalar linear and semilinear problems restricts the validity of the theoretical results to the class of semilinear problems described in §2. In future work we will consider the case of coupled systems of scalar semilinear PDE from this class, as well as more general nonlinear elliptic systems.

To simplify the presentation and keep the paper focused, we assumed that the triangulations resolve the interface. For general interface Γ, namely, Γ can not be resolved by the triangulation, we could use the concept of "δ-resolved triangulation" (cf. [21]). The results in this article could be generalized in a direct way if the triangulation satisfies the "δ-resolved". Without significant technical modifications to the results in the article, we could also relax the Local Monotonicity Assumption 3.4 to the following:

$$b'(\xi) > -\lambda_1,$$

where λ_1 is the smallest eigenvalue of the operator $-\nabla \cdot (D\nabla \cdot)$.

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