DETERMINING PROJECTIONS AND FUNCTIONALS FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

MICHAEL HOLST AND EDRISS TITI

ABSTRACT. In this paper we prove that an operator which projects weak solutions of the two- or three-dimensional Navier-Stokes equations onto a finite-dimensional space is determining if it annihilates the difference of two "nearby" weak solutions asymptotically, and if it satisfies a single appoximation inequality. We then apply this result to show that the long-time behavior of weak solutions to the Navier-Stokes equations, in both two- and three-dimensions, is determined by the long-time behavior of a finite set of bounded linear functionals. These functionals are constructed by local surface averages of solutions over certain simplex volume elements, and are therefore well-defined for weak solutions. Moreover, these functionals define a projection operator which satisfies the necessary approximation inequality for our theory. We use the general theory to establish lower bounds on the simplex diameters in both two- and three-dimensions. Furthermore, in the three dimensional case we make a connection between their diameters and the Kolmogoroff dissipation small scale in turbulent flows.

CONTENTS

1. INTRODUCTION

Consider a viscous incompressible fluid in $\Omega \subset \mathbb{R}^d$, where Ω is an open bounded domain with Lipschitz continuous boundary, and where $d = 2$ or $d = 3$. Given the kinematic viscosity $\nu > 0$, and the vector volume force function $f(x,t)$ for each $x \in \Omega$ and $t \in (0,\infty)$, the governing Navier-Stokes equations for the fluid velocity vector $u = u(x,t)$ and the scalar pressure field $p = p(x,t)$ are:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad in \ \ \Omega \times (0,\infty), \tag{1.1}$$

$$\nabla \cdot u = 0 \quad in \ \ \Omega \times (0,\infty). \tag{1.2}$$

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Also provided are initial conditions $u(0) = u_0$, as well as appropriate boundary conditions on $\partial\Omega \times (0, \infty)$.

The notion of *determining modes* for the Navier-Stokes equations was first introduced in [13] as an attempt to identify and estimate the number of degrees of freedom in turbulent flows (cf. [9] for a thorough discussion of the role of determining sets in turbulence theory). This concept later led to the notion of *Inertial Manifolds* [14]. An estimate of the number of determining modes was given in [12] and later improved in [21]. The notion of *determining nodes*, and other more general determining concepts, were introduced in [15]. In [16] the notion of determining nodes was discussed in detail, and estimates for their number were reported in [20], and later improved in [21]. In [17] (see also [19]) the concept of *determining volume elements* was presented, and a connection was established between this concept and Inertial Manifolds. A generalized and unified theory of all of the above was recently presented in [5, 6].

Bounds on the number of determining modes, nodes, and volumes are usually phrased in terms of a generalized *Grashof number*, which is defined for the two-dimensional Navier-Stokes equations as:

$$Gr = \frac{\rho^2 F}{\nu^2} = \frac{F}{\lambda_1 \nu^2},$$

where λ_1 is the smallest eigenvalue of the Stokes operator and $\rho = \sqrt{\lambda_1}$ is the related (best) Poincaré constant. Here, $F = \limsup_{t\to\infty} (\int_\Omega |f(x,t)|^2)^{1/2}$ if $f \in L^2(\Omega)$ for almost every t, or $F = \limsup_{t\to\infty} \sqrt{\lambda_1} \|f\|_{H^{-1}(\Omega)}$ if $f \in H^{-1}(\Omega)$ for almost every t.

The best known estimate for the determining set size for the two-dimensional Navier-Stokes equations with periodic boundary conditions and H^2-regular solutions is of order Gr [21]. In obtaining their estimate, the authors relied on the fact that the domain had no physical boundaries to shed vorticity, which made available some convenient properties of H^2-regular solutions. However, in the two-dimensional case with no-slip boundary conditions, to our knowledge the best estimate on the cardinal of any determining set (modes, nodes, or volumes) that can be obtained is of order Gr^2, even for H^2-regular solutions.

Due to the Sobolev Imbedding Theorem $H^2 \hookrightarrow C^0$ (which holds in dimensions 1, 2, and 3), or rather due to the failure of the imbedding $H^1 \hookrightarrow C^0$ in dimensions 2 and 3, determining node analysis is necessarily restricted to H^2-regular solutions to make sense of point-wise values. However, when talking about determining modes or volume elements, it is sufficient for functions to be H^1-regular, so that these concepts also make sense for weaker solutions. To construct a general analysis framework for the case of weak H^1 solutions, we can begin by defining notions of *determining projections* and *determining functionals* for weak solutions. (The standard spaces H, V, and V' are reviewed fully in §2.)

Definition 1.1. Let $f(t), g(t) \in V'$ be any two forcing functions satisfying

$$\lim_{t\to\infty} \|f(t) - g(t)\|_{V'} = 0, \tag{1.3}$$

and let $u, v \in V$ be corresponding weak solutions to (1.1)–(1.2). The projection operator $R_N : V \mapsto V_N \subset L^2(\Omega)$, $N = \dim(V_N) < \infty$, is called a *determining projection* for weak solutions of the d-dimensional Navier-Stokes equations if

$$\lim_{t\to\infty} \|R_N(u(t) - v(t))\|_{L^2(\Omega)} = 0, \tag{1.4}$$

implies that

$$\lim_{t\to\infty} \|u(t) - v(t)\|_H = 0. \tag{1.5}$$

Given a basis $\{\phi_i\}_{i=1}^N$ for the finite-dimensional space V_N, and a set of bounded linear functionals $\{l_i\}_{i=1}^N$ from V', we can construct a projection operator as:

$$R_N u = \sum_{i=1}^{N} l_i(u)\phi_i. \tag{1.6}$$

The assumption (1.4) is then implied by:

$$\lim_{t\to\infty} |l_i(u(t) - v(t))| = 0, \qquad i = 1, \dots, N \tag{1.7}$$

so that we can ask equivalently whether the set $\{l_i\}_{i=1}^N$ forms a set of *determining functionals* (see [5, 6]). The analysis of whether R_N or $\{l_i\}_{i=1}^N$ are determining can be reduced to an analysis of the approximation properties of R_N. Note that in this construction, the basis $\{\phi_i\}_{i=1}^N$ need not span a subspace of the solution space V, so that the functions ϕ_i need not be divergence-free for example. Note that Definition 1.1 encompasses each of the notions of determining modes, nodes, and volumes by making particular choices for the sets of functions $\{\phi_i\}_{i=1}^N$ and $\{l_i\}_{i=1}^N$ (see [19, 20]).

In this paper, we will employ Definition 1.1 to extend the results of [5, 6] to the more general setting of H^1-regular solutions. In particular, we will show that if a projection operator $R_N : V \mapsto V_N \subset L^2(\Omega)$, $N = \dim(V_N) < \infty$, satisfies an approximation inequality for $\gamma > 0$ of the form,

$$\|u - R_N u\|_{L^2(\Omega)} \le C_1 N^{-\gamma} \|u\|_{H^1(\Omega)}, \tag{1.8}$$

then the operator R_N is a determining projection in the sense of Definition 1.1, provided N is large enough. We will also derive explicit bounds on the dimension N which guarantees that R_N is determining. While we gain generality in our approach here, we also lose something in the balance: the bounds obtained here are generally of order Gr^2, whereas the bounds in [5, 6] (requiring H^2-regularity) are of order Gr.

Outline of the paper. Preliminary material is presented in §2, including some inequalities for bounding the nonlinear term appearing in weak formulations of the Navier-Stokes equations. In §3, a finite element interpolant due to Scott and Zhang is presented, which (unlike nodal interpolation) is well-defined for H^1-functions. It is shown that the interpolant satisfies the approximation assumption (1.8) for H^1-functions on arbitrary polyhedral domains in both two and three dimensions; most of the details are relegated to the Appendix. In §4, we consider the two-dimensional Navier-Stokes equations, and derive bounds on the dimension N of the space V_N, employing only the approximation assumption (1.8). As an application of this general result, we employ some standard assumptions about simplex triangulations of the domain (discussed in §3) and derive lower bounds on the simplex diameters, sufficient to ensure that the SZ-interpolant is a determining projection (equivalently, that the simplex surface integrals forming SZ-interpolant coefficients are a determining set of linear functionals). We extend these results to three dimensions in §5, by requiring (following [7]) that weak solutions satisfy an additional technical assumption (due to the lack of appropriate global *a priori* estimates), which is related to the natural notion of mean dissipation rate of energy.

2. Preliminary Material

We briefly review some background material following the notation of [8, 23, 25, 26]. Let $\Omega \subset \mathbb{R}^d$ denote an open bounded set. The imbedding results we will need are known to hold for example if the domain Ω has a locally Lipschitz boundary, denoted as $\Omega \in \mathcal{C}^{0,1}$ (cf. [1]). For example, open bounded convex sets $\Omega \subset \mathbb{R}^d$ satisfy $\Omega \in \mathcal{C}^{0,1}$

(Corollary 1.2.2.3 in [18]), so that convex polyhedral domains (which we consider here) are in $\mathcal{C}^{0,1}$.

Let $H^k(\Omega)$ denote the usual Sobolev spaces $W^{k,2}(\Omega)$. Employing multi-index notation, the distributional partial derivative of order $|\alpha|$ is denoted D^α, so that the (integer-order) norms and semi-norms in $H^k(\Omega)$ may be denoted

$$\|u\|^2_{H^k(\Omega)} = \sum_{j=0}^{k} |\Omega|^{\frac{j-k}{d}} |u|^2_{H^j(\Omega)}, \qquad |u|^2_{H^j(\Omega)} = \sum_{|\alpha|=j} \|D^\alpha u\|_{L^2(\Omega)}, \quad 0 \le j \le k,$$

where $|\Omega|$ represents the measure of Ω. Fractional order Sobolev spaces and norms may be defined for example through Fourier transform and extension theorems, or through interpolation. A fundamentally important subspace is the $k = 1$ case of

$$H^k_0(\Omega) = closure\ of\ C^\infty_0(\Omega) \text{ in } H^k(\Omega),$$

in which the Poincaré Inequality reduces to: If Ω is bounded, then

$$\|u\|_{L^2(\Omega)} \le \rho(\Omega)|u|_{H^1(\Omega)}, \qquad \forall u \in H^1_0(\Omega). \tag{2.1}$$

The spaces above extend naturally (cf. [25]) to product spaces of vector functions $u = (u_1, u_2, \ldots, u_d)$, which are denoted with the same letters but in bold-face; for example, $\mathbf{H}^k_0(\Omega) = \left(H^k_0(\Omega)\right)^d$. The inner-products and norms in these product spaces are extended in the natural Euclidean way; the convention here will be to subscript these extended vector norms the same as the scalar case.

Define now the space $\mathcal{V}$ of divergence free $\mathbf{C}^\infty$ vector functions with compact support as

$$\mathcal{V} = \{\phi \in \mathbf{C}^\infty_0(\Omega) \mid \nabla \cdot \phi = 0\}.$$

The following two subspaces of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1_0(\Omega)$ are fundamental to the study of the Navier-Stokes equations.

$$H = closure\ of\ \mathcal{V} \text{ in } \mathbf{L}^2(\Omega), \qquad V = closure\ of\ \mathcal{V} \text{ in } \mathbf{H}^1_0(\Omega).$$

To simplify the notation, it is common (cf. [8, 25]) to use the following notation for inner-products and norms in H and V:

$$(u, v) = (u, v)_H, \quad |u| = \|u\|_H, \quad ((u, v)) = (u, v)_V, \quad \|u\| = \|u\|_V. \tag{2.2}$$

The Navier-stokes equations (1.1)–(1.2) are equivalent to the functional differential equation:

$$\frac{du}{dt} + \nu A u + B(u, u) = f, \qquad u(0) = u_0. \tag{2.3}$$

The Stokes operator A and bilinear form B are defined as

$$Au = -P\Delta u, \qquad B(u, v) = P[(u \cdot \nabla)v],$$

where the operator P is the Leray orthogonal projector, $P : H^1_0 \mapsto V$ and $P : L^2 \mapsto H$, respectively.

Weak formulations, which we consider shortly, will use the bilinear Dirichlet form $((\cdot, \cdot))$ and trilinear form $b(\cdot, \cdot, \cdot)$ as:

$$((u, v)) = (\nabla u, \nabla v), \qquad b(u, v, w) = (B(u, v), w) = (P((u \cdot \nabla)v), w).$$

(Note that thanks to the Poincaré inequality (2.1), the form $((\cdot, \cdot))$ is actually an inner-product on V, and the induced norm $\|\cdot\| = ((\cdot, \cdot))^{1/2}$ is in fact a norm on V, equivalent to the H^1-norm.)

A priori bounds can be derived for the form $b(\cdot,\cdot,\cdot)$ (cf. [8, 22, 25]). In particular, if $\Omega \subset \mathbb{R}^d$, then the trilinear form $b(u,v,w)$ is bounded on $V \times V \times V$ as follows:

$$d = 2: \quad |b(u,v,w)| \le 2^{1/2}\|u\|_{L^2(\Omega)}^{1/2}|u|_{H^1(\Omega)}^{1/2}|v|_{H^1(\Omega)}\|w\|_{L^2(\Omega)}^{1/2}|w|_{H^1(\Omega)}^{1/2}, \tag{2.4}$$

$$d = 3: \quad |b(u,v,w)| \le 2\|u\|_{L^2(\Omega)}^{1/4}|u|_{H^1(\Omega)}^{3/4}|v|_{H^1(\Omega)}\|w\|_{L^2(\Omega)}^{1/4}|w|_{H^1(\Omega)}^{3/4}. \tag{2.5}$$

Moreover, from Hölder inequalities we have for $d = 2$ or $d = 3$:

$$|b(v,u,v)| \le \|\nabla u\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}^2. \tag{2.6}$$

3. Polynomial interpolation in $\mathbf{H}_0^1(\Omega)$

An example of a projection operator which satisfies the approximation assumption (1.8) is that used for defining determining volumes [19]; we examine now powerful alternative operator. Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional polygon, exactly triangulated by (for example) Delaunay triangulation [11], with quasi-uniform, shape-regular simplices, the vertices of which will form a set of N generalized interpolation points in our analysis. Note that for quasi-uniform, shape-regular triangulations in $\mathbb{R}^d$ (see [4] for detailed discussions), it holds that

$$C_0|\Omega|h^{-d} \le N \le C_0'|\Omega|h^{-d}, \tag{3.1}$$

where h is the maximum of the diameters of the simplices, and where C_0 and C_0' are universal constants, independent of both N and h. The parameter h will be referred to as the characteristic parameter, or characteristic length scale, of such a quasi-uniform shape-regular mesh.

It should be noted that given some initial triangulation satisfying (3.1), repeated bisection [2] or octa-section [27] (quadra-section in 2D) of each simplex can be performed in such a way as to guarantee non-degeneracy asymptotically, in that the quasi-uniformity and shape-regularity are preserved. Therefore, inequality (3.1) can be made to hold, for the same universal constants, for finer and finer meshes in a nested sequence of simplex triangulations.

To properly define a continuous piecewise-linear nodal interpolant of a function $u \in H^1(\Omega)$ based on the nodes of a triangulation of Ω, the particular function u must be bounded point-wise. This will be true if the function u is continuous in Ω, hence uniformly continuous on $\bar{\Omega}$. One of the Sobolev imbedding results (cf. [1]) states that if $\Omega \subset \mathbb{R}^d$ satisfies $\Omega \in \mathcal{C}^{0,1}$, then for nonnegative real numbers k and s it holds that $H^k(\Omega) \hookrightarrow C^s(\bar{\Omega})$, $k > s + \frac{d}{2}$. This implies that for $d = 1$, the interpolant can be correctly defined, since $H^1(\Omega)$ is continuously imbedded in $C^0(\bar{\Omega})$. However, in higher dimensions, $H^{1+\alpha}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ only if $\alpha > 0$ when $d = 2$, or if $\alpha > 1/2$ when $d = 3$. While it may be possible to use the nodal interpolant and a regularity assumption such as $u \in H^{1+\alpha}(\Omega)$ for appropriate $\alpha > 0$, an alternative approach is taken here.

The generalized interpolant due to Scott and Zhang [24] can be defined for H^1-functions in both two and three spatial dimensions. The SZ-interpolant I_h is constructed from a combination linear interpolation and local averaging on faces and edges of simplices, and has optimal approximation properties even in the case of H^1-functions.

Lemma 3.1. *For the SZ-interpolant of $u \in \mathbf{H}_0^{1+\alpha}(\Omega)$, $\alpha \ge 0$, it holds that*

$$\|u - I_h u\|_{L^2(\Omega)} \le C_1 h^{1+\alpha}|u|_{H^{1+\alpha}(\Omega)}.$$

Proof. See the appendix for a condensed proof following [3, 24]. □

Note that both the usual nodal interpolant and the SZ-interpolant I_h can be written as a linear combination of linear functionals:

$$I_h u(x) = \sum_{i=1}^{N} \phi_i(x) l_i(u).$$

In either case, the set of functions $\{\phi_i\}_{i=1}^N$ is the usual continuous piecewise-polynomial nodal finite element basis defined over the simplicial mesh, satisfying the *Lagrange* property at the vertices of the mesh:

$$\phi_i(x_j) = \delta_{ij}.$$

The difference between the two interpolants is simply the choice of the linear functionals: in the case of the nodal interpolant, the functionals are delta functions centered at the vertices of the mesh; in the case of the SZ-interpolant, they are defined in terms of a bi-orthogonal dual basis (see the Appendix).

4. The Two-dimensional Navier-Stokes Equations

A general weak formulation of the Navier-Stokes equations (1.1)–(1.2) can be written as (cf. [8, 25]):

Definition 4.1. Given $f \in L^2([0,T]; V')$, a weak solution of the Navier-Stokes equations satisfies $u \in L^2([0,T]; V) \cap C_w([0,T]; H)$, $du/dt \in L^1_{\text{loc}}((0,T]; V')$, and

$$< \frac{du}{dt}, v > + \nu((u,v)) + b(u,u,v) =< f, v >, \quad \forall v \in V, \quad for\ almost\ every\ t, \quad (4.1)$$

$$u(0) = u_0. \quad (4.2)$$

Here, the space $C_w([0,T]; H)$ is the subspace of $L^\infty([0,T]; H)$ of weakly continuous functions, and $< \cdot, \cdot >$ denotes the duality pairing between V and V', where H is the Riesz-identified pivot space in the Gelfand triple $V \subset H = H' \subset V'$. Note that since the Stokes operator can be uniquely extended to $A : V \mapsto V'$, and since it can be shown that $B : V \times V \mapsto V'$ (cf. [8, 26] for both results), the functional form (2.3) still makes sense for weak solutions, and the total operator represents a mapping $V \mapsto V'$.

In the two-dimensional case, for a forcing function $f \in L^\infty([0,T]; V')$, there exists a unique weak solution $u \in L^2([0,T]; V) \cap C_w([0,T]; H)$ (cf. [8, 26]). Consider now two forcing functions $f, g \in L^2([0,\infty]; V')$ and corresponding weak solutions u and v to (2.3) in either the two- or three-dimensional case. Subtracting the equations (2.3) for u and v yields an equation for the difference function $w = u - v$, namely

$$\frac{dw}{dt} + \nu A w + B(u,u) - B(v,v) = f - g. \quad (4.3)$$

Since the residual of equation (4.3) lies in the dual space V', for almost every t, we can consider the dual pairing of each side (4.3) with a function in V, and in particular with $w \in V$, which yields

$$< \frac{dw}{dt}, w > + \nu \|w\|^2 + b(u,u,w) - b(v,v,w) =< f - g, w > \quad for\ almost\ every\ t.$$

It can be shown (cf. [25], Chapter 3, Lemma 1.2) that

$$\frac{1}{2}\frac{d}{dt}|w|^2 =< \frac{dw}{dt}, w >$$

in the distribution sense. It can also be shown [8, 25] that $b(u,v,w) = -b(u,w,v)$, $\forall u,v,w \in V$, so that $b(w,u,w) = b(u,u,w) - b(v,v,w)$. Therefore, the function $w = u - v$ must satisfy

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + b(w,u,w) =< f-g, w > . \tag{4.4}$$

The following generalized Gronwall inequality will be a key tool in the analysis to follow (see [12] and [19]).

Lemma 4.2. *Let $T > 0$ be fixed, and let $\alpha(t)$ and $\beta(t)$ be locally integrable and real-valued on $(0,\infty)$, satisfying:*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)d\tau = m > 0, \qquad \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)d\tau = M < \infty,$$

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)d\tau = 0,$$

where $\alpha^- = \max\{-\alpha, 0\}$ and $\beta^+ = \max\{\beta, 0\}$. If $y(t)$ is an absolutely continuous non-negative function on $(0,\infty)$, and $y(t)$ satisfies the following differential inequality:

$$y'(t) + \alpha(t)y(t) \le \beta(t), \ \text{ a.e. on } (0,\infty),$$

then $\lim_{t\to\infty} y(t) = 0$.

The main two-dimensional results are now given; we assume that $\Omega \subset \mathbb{R}^2$ is an open bounded domain with Lipschitz continuous boundary.

Theorem 4.3. *Let $f(t), g(t) \in V'$ be any two forcing functions satisfying*

$$\lim_{t\to\infty} \|f(t) - g(t)\|_{V'} = 0,$$

and let $u, v \in V$ be the corresponding weak solutions to (1.1)–(1.2) for $d = 2$. If there exists a projection operator $R_N : V \mapsto V_N \subset L^2(\Omega)$, $N = \dim(V_N)$, satisfying

$$\lim_{t\to\infty} \|R_N(u(t) - v(t))\|_{L^2(\Omega)} = 0,$$

and satisfying for $\gamma > 0$ the approximation inequality

$$\|u - R_N u\|_{L^2(\Omega)} \le C_1 N^{-\gamma}\|u\|_{H^1(\Omega)},$$

then

$$\lim_{t\to\infty} |u(t) - v(t)| = 0$$

holds if N is such that

$$\infty > N > C\left(\frac{1}{\nu^2}\limsup_{t\to\infty}\|f(t)\|_{V'}\right)^{\frac{1}{\gamma}},$$

where C is a constant independent of ν and f.

Proof. Using the notation (2.2), we begin with equation (4.4), employing the inequality (2.4) along with Cauchy-Schwarz and Young's inequalities to yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 &\le \|u\|\;|w|\;\|w\| + \|f-g\|_{V'}\|w\| \\ &\le \frac{1}{\nu}\|u\|^2|w|^2 + \frac{1}{\nu}\|f-g\|_{V'}^2 + \frac{\nu}{2}\|w\|^2.\end{aligned}$$

Equivalently, this is

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 - \frac{2}{\nu}\|u\|^2|w|^2 \leq \frac{2}{\nu}\|f-g\|_{V'}^2.$$

To bound the second term on the left from below, we employ the approximation assumption on R_N, or rather the following inequality which follows from it:

$$|w|^2 \leq 2N^{-2\gamma}C_1^2\|w\|^2 + 2\|R_N w\|_{L^2(\Omega)}^2,$$

which yields

$$\frac{d}{dt}|w|^2 + \left(\frac{\nu N^{2\gamma}}{2C_1^2} - \frac{2}{\nu}\|u\|^2\right)|w|^2 \leq \frac{2}{\nu}\|f-g\|_{V'}^2 + \frac{\nu N^{2\gamma}}{C_1^2}\|R_N w\|_{L^2(\Omega)}^2.$$

This is of the form

$$\frac{d}{dt}|w|^2 + \alpha|w|^2 \leq \beta,$$

with obvious definition of α and β.

The generalized Gronwall Lemma 4.2 can now be applied. Recall that both $\|f - g\|_{V'} \to 0$ and $\|R_N w\|_{L^2(\Omega)} \to 0$ as $t \to \infty$ by assumption. Since it is assumed that u and v, and hence w, are in V, so that all other terms appearing in α and β remain bounded, it must hold that

$$\lim_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta^+(\tau)d\tau = 0, \qquad \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha^-(\tau)d\tau < \infty.$$

It remains to verify that for some fixed $T > 0$,

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau > 0.$$

This means we must verify the following inequality for some fixed $T > 0$:

$$N^{2\gamma} > \frac{2C_1^2}{\nu}\left(\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{2\|u\|^2}{\nu}d\tau\right) = \frac{4C_1^2}{\nu^2}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u\|^2 d\tau. \tag{4.5}$$

The following *a priori* bound on any weak solution can be shown to hold (this is a simple generalization to $f \in V'$ of the bound in [8] for $f \in H$):

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u(\tau)\|^2 d\tau \leq \frac{2}{\nu^2}\limsup_{t\to\infty}\|f(t)\|_{V'}^2,$$

for $T = \rho^2/\nu > 0$, where ρ is the best constant from the Poincaré inequality (2.1). Therefore, if

$$N^{2\gamma} > 8C_1^2\left(\frac{1}{\nu^2}\limsup_{t\to\infty}\|f(t)\|_{V'}\right)^2 \geq \frac{4C_1^2}{\nu^2}\left(\frac{2}{\nu^2}\limsup_{t\to\infty}\|f(t)\|_{V'}^2\right), \tag{4.6}$$

implying that (4.5) holds, then by the Gronwall Lemma 4.2, it follows that

$$\lim_{t\to\infty}|w(t)| = \lim_{t\to\infty}|u(t) - v(t)| = 0.$$

□

Assume now that $\Omega \subset \mathbb{R}^2$ is also polyhedral, and can be exactly triangulated with a quasi-uniform, shape-regular set of simplices of maximal diameter $h = O(N^{-1/2})$, where N is the number of vertices in the triangulation (see §3). As an application of the general result above, we establish a lower bound on the simplex diameters of such a triangulation, which ensures that the SZ-interpolant is a determining projection (equivalently, that the simplex surface integrals forming SZ-interpolant coefficients are a determining set of linear functionals).

Corollary 4.4. *The SZ-interpolant is determining for the two-dimensional Navier-Stokes equations if the diameter h of the simplices is small enough so that*

$$\infty > h^{-2} > C\left(\frac{1}{\nu^2}\limsup_{t\to\infty}\|f(t)\|_{V'}\right)^2.$$

Proof. Since $h = O(N^{-1/2})$ for quasi-uniform, shape-regular triangulations in two dimensions, taking $\alpha = 0$ in Lemma 3.1 yields

$$\|u - I_h u\|_{L^2(\Omega)} \leq C_1 h|u|_{H^1(\Omega)} \leq \tilde{C}_1 N^{-1/2}\|u\|_{H^1(\Omega)}.$$

Therefore, the SZ-interpolant I_h satisfies the approximation inequality (1.8) for $\gamma = 1/2$. The corollary then follows by application of Theorem 4.3. □

Remark 4.5. *If $f \in H$, then we have in fact a strong solution, i.e. $u \in H^2(\Omega)$, and the interpolation Lemma 3.1 may be applied with $\alpha = 1$. This falls into the theoretical framework of* [5, 6]*, and in the periodic case they have shown that $N \approx Gr$, whereas the above result for the no-slip case states that $N \approx Gr^2$. Whether the no-slip case may be improved to $N \approx Gr$ with additional regularity ($f \in H$) is unclear, due to the lack of an analogous identity to*

$$(B(w, w), Aw) = 0,$$

which holds for the two-dimensional periodic case. In physical terms, in two dimensions this identity illustrates the lack of a boundary vorticity shedding source when the boundary is absent.

5. The Three-dimensional Navier-Stokes Equations

The lack of appropriate *a priori* estimates in the three-dimensional case requires a modification of the approach taken for the two-dimensional case in the previous section. However, the interpolation results we have employed are dimension-independent, and by following the analysis approach of [7] very closely, we can obtain similar results for the three-dimensional case. Again we require only that $f \in V'$, but we also assume the existence of a unique weak solution to the three-dimensional Navier-Stokes equations. An additional technical assumption is that some measure of the mean rate of energy dissipation be finite, namely:

$$\epsilon_\infty = \inf_{T>0}\limsup_{t\to\infty}\frac{\nu}{T}\int_t^{t+T}\|\nabla u\|_\infty d\tau < \infty.$$

This assumption implies that eventually the weak solution for the three-dimensional Navier-Stokes equations becomes unique, and also in the case $f \in H$ the weak solution eventually becomes strong. But this assumption does not imply anything about the transients, since the quantity is required to be finite only for large time. We assume again that $\Omega \subset \mathbb{R}^3$ is an open bounded domain with Lipschitz continuous boundary.

Theorem 5.1. *Let $f(t), g(t) \in V'$ be any two forcing functions satisfying*

$$\lim_{t\to\infty}\|f(t) - g(t)\|_{V'} = 0,$$

and let $u, v \in V$ be the corresponding weak solutions to (1.1)–(1.2) for $d = 3$. If there exists a projection operator $R_N : V \mapsto V_N \subset L^2(\Omega)$, $N = \dim(V_N)$, satisfying

$$\lim_{t\to\infty}\|R_N(u(t) - v(t))\|_{L^2(\Omega)} = 0,$$

and satisfying for $\gamma > 0$ the approximation inequality

$$\|u - R_N u\|_{L^2(\Omega)} \leq C_1 N^{-\gamma} \|u\|_{H^1(\Omega)},$$

then

$$\lim_{t\to\infty} |u(t) - v(t)| = 0$$

holds if N is such that

$$\infty > N > C\left(\frac{1}{\nu}\inf_{T>0}\left\{\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\nabla u(s)\|_{L^\infty(\Omega)}ds\right\}\right)^{\frac{1}{2\gamma}},$$

where C is a constant independent of ν, f, and u.

Proof. Beginning with equation (4.4), the inequality (2.6) is employed along with Cauchy-Schwarz and Young's inequalities to yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 &\leq \|\nabla u\|_{L^\infty(\Omega)}|w|^2 + \|f-g\|_{V'}\|w\| \\ &\leq \|\nabla u\|_{L^\infty}|w|^2 + \frac{1}{2\nu}\|f-g\|_{V'}^2 + \frac{\nu}{2}\|w\|^2\end{aligned}$$

Equivalently,

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 - \|\nabla u\|_{L^\infty(\Omega)}|w|^2 \leq \frac{1}{\nu}\|f-g\|_{V'}^2.$$

To bound the second term on the left from below, we employ a consequence of the approximation assumption on R_N, namely the inequality

$$|w|^2 \leq 2N^{-2\gamma}C_1^2\|w\|^2 + 2\|R_N w\|_{L^2(\Omega)}^2,$$

which yields

$$\frac{d}{dt}|w|^2 + \left(\frac{\nu N^{2\gamma}}{2C_1^2} - \|\nabla u\|_{L^\infty}\right)|w|^2 \leq \frac{1}{\nu}\|f-g\|_{V'}^2 + \frac{\nu N^{2\gamma}}{C_1^2}\|R_N w\|_{L^2(\Omega)}^2.$$

This has the form

$$\frac{d}{dt}|w|^2 + \alpha|w|^2 \leq \beta,$$

with again obvious definition of α and β.

The analysis now proceeds exactly as in the proof of Theorem 4.3, so that all that remains is to check again that for some fixed $T > 0$,

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau > 0.$$

Thus, we must prove our assumption on N guarantees for a fixed $T > 0$ that

$$N^{2\gamma} > \frac{2C_1^2}{\nu}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\nabla u\|_{L^\infty(\Omega)}d\tau. \tag{5.1}$$

If we select $T_* > 0$ such that

$$2\inf_{T>0}\left(\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\nabla u(s)\|_{L^\infty(\Omega)}ds\right) \geq \limsup_{t\to\infty}\frac{1}{T_*}\int_t^{t+T_*}\|\nabla u(s)\|_{L^\infty(\Omega)}ds,$$

then our assumption gives

$$N^{2\gamma} > \frac{4C_1^2}{\nu}\inf_{T_*>0}\left(\limsup_{t\to\infty}\frac{1}{T_*}\int_t^{t+T_*}\|\nabla u(s)\|_{L^\infty(\Omega)}ds\right) \tag{5.2}$$

which implies (5.1). The theorem then follows by the Gronwall Lemma 4.2. $\square$

Assume now that $\Omega \subset \mathbb{R}^3$ is also polyhedral, and can be exactly triangulated with a quasi-uniform, shape-regular set of simplices of maximal diameter $h = O(N^{-1/3})$, where N is the number of vertices in the triangulation. As an application of the general three-dimensional result above, we will establish a lower bound on the simplex diameters of such a triangulation, which ensures that the SZ-interpolant is a determining projection (and that the simplex surface integrals forming SZ-interpolant coefficients are a determining set of linear functionals).

Corollary 5.2. *The SZ-interpolant is determining for the three-dimensional Navier-Stokes equations if the diameter h of the simplices is small enough so that*

$$\infty > h^{-2} > C\left(\frac{1}{\nu}\inf_{T>0}\left\{\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\nabla u(s)\|_{L^\infty(\Omega)}ds\right\}\right).$$

Proof. Since $h = O(N^{-1/3})$ for quasi-uniform, shape-regular triangulations in three dimensions, taking $\alpha = 0$ in Lemma 3.1 yields

$$\|u - I_h u\|_{L^2(\Omega)} \le C_1 h|u|_{H^1(\Omega)} \le \tilde{C}_1 N^{-1/3}\|u\|_{H^1(\Omega)}.$$

Therefore, the SZ-interpolant I_h satisfies the approximation inequality (1.8) for $\gamma = 1/3$. The corollary then follows by application of Theorem 5.1. □

APPENDIX: APPROXIMABILITY OF THE SCOTT-ZHANG INTERPOLANT

We will sketch the proof of the approximability result for the SZ-interpolant given as Lemma 3.1; we will follow quite closely the proof given in [3, 24]. As throughout this paper, we assume that $\Omega \in \mathcal{C}^{0,1}$, and that the given exact simplicial triangulation of Ω is both shape-regular and quasi-uniform.

The proof of Lemma 3.1 will follow easily from the following result (see the comments at the end of this appendix).

Lemma 5.3. *For the SZ-interpolant of $u \in H_0^{1+\alpha}(\Omega)$, $\alpha \ge 0$, it holds that*

$$\|u - I_h u\|_{L^2(\Omega)} \le C_1 h^{1+\alpha}|u|_{H^{1+\alpha}(\Omega)}.$$

To prove Lemma 5.3, we will begin by defining carefully the SZ-interpolant. Let $\mathcal{T}_h = \{\tau_i\}_{i=1}^L$ be the given quasi-uniform, shape-regular mesh of d-simplices which exactly triangulate the underlying domain Ω, and let $\Omega_h = \{x_i\}_{i=1}^N$ be the set of vertices of these d-simplices. Define

$$V_h = \text{span}\{\phi_i(x)\}_{i=1}^N \subset H^1(\Omega),$$

where $\{\phi_i(x)\}$ is the set of standard continuous piecewise linear (nodal) basis functions. The nodal basis satisfies the Lagrange relationship at the vertices (which are exactly the "nodes" in this setting):

$$\phi_i(x_j) = \delta_{ij}.$$

Now, for each vertex x_i, we select (arbitrarily) an associated $(d-1)$-simplex σ_i from the given simplicial mesh satisfying only:

$$(1)\ x_i \in \bar{\sigma}_i, \quad \text{and} \quad (2)\ \sigma_i \subset \partial\Omega \text{ if } x_i \in \partial\Omega.$$

In other words, for a given vertex x_i we pick an arbitrary $(d-1)$-simplex from edges or faces of the d-simplices which contain x_i as a vertex. In two-dimensions, we are picking the edge of one of the triangles that have x_i as a vertex; in three-dimensions, we are picking the face of one of the tetrahedra which have x_i as a vertex. The only restriction on this choice is near the boundary: if x_i is on the boundary, then the $(d-1)$-simplex we pick must be one of the edges or faces of the a simplex which lies exactly on the boundary (such a choice is always possible).

In each $(d-1)$-simplex σ_i, we number the generating vertex x_i first in the set of vertices of σ_i, denoted $\{x_{i,j}\}_{j=1}^d$. (I.e., we set $x_{i,1} = x_i$.) For each σ_i, we also have a $(d-1)$-dimensional nodal basis $\{\phi_{i,j}\}_{j=1}^d$, where again we set $\phi_{i,1} = \phi_i$. There exists an associated $L^2(\sigma_i)$-dual (bi-orthogonal) basis $\{\psi_{i,j}\}$ satisfying

$$\int_{\sigma_i} \psi_{i,j}(x)\phi_{i,k}(x)dx = \delta_{jk}, \quad j,k = 1,\ldots,d.$$

Again we take $\psi_{i,1} = \psi_i$, $\forall x_i \in \Omega_h$. Note that ψ_i and ϕ_j also satisfy a bi-orthogonal relationship, namely $\int_{\sigma_i} \psi_i\phi_j dx = 0$, $i \neq j$. We define now the SZ-interpolant as

$$I_h : H^1(\Omega) \mapsto V_h(\Omega), \quad I_h u(x) = \sum_{i=1}^N \phi_i(x) l_i(u), \quad l_i(u) = \int_{\sigma_i} \psi_i(\xi)u(\xi)d\xi.$$

Thanks to the Trace Theorem [1], the interpolant $I_h u(x)$ is well-defined at nodal values even for $u \in H^1(\Omega)$, since $H^1(\Omega) \hookrightarrow L^2(\sigma_i)$. Almost by construction, one can show [24] that

- $I_h : H^1(\Omega) \mapsto V_h(\Omega)$ is a projection
- $I_h : H_0^1(\Omega) \mapsto V_{0h}(\Omega)$

where V_{0h} is the subset of V_h having zero trace on the boundary of Ω. Thus, I_h preserves homogeneous Dirichlet boundary conditions. Using homogeneity arguments, the following stability result for the interpolant is established in [24].

Lemma 5.4. *For any $\tau \in \mathcal{T}_h$, if the support region of τ is defined as the set $S_\tau = interior\ (\cup\{\bar{\tau}_i \mid \bar{\tau}_i \cap \bar{\tau} \neq \emptyset,\ \tau_i \in \mathcal{T}_h\})$, then it holds that*

$$\|I_h u\|_{H^m(\tau)} \leq C \sum_{k=0}^{l} h^{k-m} |u|_{H^k(S_\tau)}, \ \ 0 \leq m \leq l, \ \ l > 1/2.$$

Proof. See the proof of Theorem 3.1 in [24]. □

The proof of the Scott and Zhang [24] approximation result is as follows.

Proof. (Lemma 5.3) Since I_h is a projector from $H^1(\Omega)$ onto $V_h(\Omega)$, it follows that on each element, I_h is a projector from $H^1(\tau)$ onto $\mathcal{P}_1(\tau)$, the space of linear polynomials over τ. Thus, $I_h p = p$, $\forall p \in \mathcal{P}_1(\tau)$, and employing also the stability result in Lemma 5.4 we have that for $0 \leq m \leq k \leq 2$,

$$\|u - I_h u\|_{H^m(\tau)} \leq \|u - p\|_{H^m(\tau)} + \|I_h(p-u)\|_{H^m(\tau)} \leq C \sum_{k=0}^{m} h^{k-m} \|u-p\|_{H^k(S_\tau)},$$

where S_τ is the element support region surrounding τ as defined in Lemma 5.4. Employing the modified Bramble-Hilbert lemma developed in [10] to estimate the terms of the sum gives

$$\inf_{p \in \mathcal{P}_1(\tau)} \|u-p\|_{H^m(S_\tau)} \leq C h^{k-m} |u|_{H^k(S_\tau)}, \ \ 0 \leq m \leq k \leq 2,$$

where due to the assumptions about the domain and the mesh, the constant C depends only on the spatial dimension d. Together with the equation above this is

$$\|u - I_h u\|_{H^m(\tau)} \leq C h^{k-m} |u|_{H^k(S_\tau)} \ \ 0 \leq m \leq k \leq 2.$$

Since the set

$$Q = \sup_{\tau \in \mathcal{T}_h} \{card\{\tau \in \mathcal{T}_h | \tau \cap S_\tau \neq \emptyset\}\}$$

is finite due to the quasi-uniformity and shape-regularity of the mesh, we have finally that for $0 \leq m \leq k \leq 2$, it holds that

$$\|u - I_h u\|^2_{H^m(\Omega)} = \sum_{\tau \in \mathcal{T}_h} \|u - I_h u\|^2_{H^m(\tau)} \leq C h^{2(k-m)} \|u\|^2_{H^k(\Omega)}.$$

The result for non-integer exponents k and m follows by the usual norm interpolation arguments between $L^2(\Omega)$ and $H^2(\Omega)$, which completes the proof. $\square$

Lemma 5.3 can be easily extended to the vector case, which provides finally the proof of Lemma 3.1.

Proof. (Lemma 3.1) For $u \in \mathbf{H}_0^{1+\alpha}(\Omega) = (H_0^{1+\alpha}(\Omega))^d$, we have that

$$\|u - I_h u\|^2_{L^2} = \sum_{i=1}^{d} \|u_i - I_h^{(i)} u_i\|_{L^2(\Omega)} \leq C_1^2 h^{2(1+\alpha)} \sum_{i=1}^{d} |u_i|^2_{H^{1+\alpha}(\Omega)},$$

where $I_h^{(i)}$ denotes the scalar SZ-interpolant applied to u_i. Thus,

$$\|u - I_h u\|_{L^2(\Omega)} \leq C_1 h^{1+\alpha} |u|_{H^{1+\alpha}(\Omega)}.$$

$\square$

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E-mail address: holst@ama.caltech.edu

APPLIED MATHEMATICS 217-50, CALTECH, PASADENA, CA 91125, USA.

DEPARTMENT OF MATHEMATICS AND DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING, UNIVERSITY OF CALIFORNIA, IRVINE, CA 92697-3875, USA.

E-mail address: etiti@math.uci.edu