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A COMPARISON OF TWO MULTILEVEL ITERATIVE METHODS FOR NONSYMMETRIC AND INDEFINITE ELLIPTIC FINITE ELEMENT EQUATIONS*

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Abstract. We describe and analyze two multilevel iterative procedures for solving linear systems arising from certain finite element discretizations of nonself-adjoint and indefinite elliptic partial differential equations. We prove convergence and optimal work estimate for both schemes.

Introduction. In this work we will discuss two strategies for solving indefinite or nonself-adjoint elliptic partial differential equations using multilevel iterative techniques applied to finite element equations. Multilevel methods have received much attention recently, primarily because many such schemes can produce sufficiently accurate solutions of discrete equations in time which, asymptotically, grows linearly in the number of unknowns [2], [3], [5], [7], [8]. Both schemes discussed in this work have this property.

Although much work in the area has focused on self-adjoint problems giving rise to symmetric and positive definite linear systems, several authors have considered the application of multilevel schemes to indefinite or nonsymmetric equations [2], [7], [9]. The general approach has been to treat the problem using essentially the same procedure as in the symmetric positive definite case and to analyze the effect of the non-symmetry or indefiniteness. The first alternative we will analyze is such an extension of the Bank-Dupont scheme presented in [3]. This scheme, as well as those of other authors, displays two features which are not present in the symmetric positive definite case:

A. There are requirements that the coarsest grid be sufficiently fine to guarantee that certain approximation and other properties hold on each of the grids. Generally, these requirements are trivially satisfied in the symmetric positive definite case. In the present case, they are often nontrivial, and failure to satisfy them can cause the scheme to diverge.

B. An undesirable link between the number of smoothing iterations and the fineness of the coarsest grid emerges. Smoothing iterations on fine grids can cause already smooth components of the error to grow; solutions of the coarse grid residual equations must sufficiently damp these growing components or divergence will occur. This is unlike the symmetric positive definite case, where the smoothing iteration can be shown to reduce the norm of the error.

A fundamental premise of multilevel schemes is that one can construct a sequence of successively refined meshes, and that the solution of a given problem on one grid can, under certain circumstances, be used to approximate the corresponding solutions on other meshes [5], [6]. The justification is that on each grid we are discretizing the same continuous elliptic operator, and our ultimate goal is the solution of some common underlying partial differential equation. Requirements A are natural consequences of this premise.

The second of these problems is the one to which our other multilevel scheme is addressed. Basically, our approach is to substitute smoothing iterations based on the normal equations for each level. One can then show that the norm of the error is

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reduced, just as in the symmetric positive definite case. The penalty associated with this alternative is that one may have to accept a somewhat slower rate of convergence for the smoothing iteration because of the effect of squaring the condition number of the linear system. This does not appear to be a serious problem in practice.

The organization of the remainder of the paper is as follows: In § 1, we state the problem, the assumptions and establish notation. Section 2 contains a description of the two multilevel schemes, theorems on their convergence rates, and work estimates. Proofs of the convergence results are given in § 3. In § 4, we present some numerical results.

1. Preliminaries. We consider the model elliptic partial differential equation

$$\begin{aligned} &\mathcal{L}u = f \quad \text{in } \Omega \subset R^2, \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$\mathcal{L}u = -\nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u + c(x)u. \tag{1.2}$$

We assume $a \in C^1(\bar{\Omega})$ and there exist positive constants $\underline{a}$ and $\bar{a}$ such that $\underline{a} \leqq a(x) \leqq \bar{a}$ for $x \in \Omega$. Further, we assume $b_1, b_2 \in C^1(\bar{\Omega})$, $c \in C^0(\bar{\Omega})$ and that $|b(x)| \leqq \bar{b}$, $|\nabla \cdot b| \leqq \bar{b}$, $|c(x)| \leqq \bar{b}$, for $x \in \Omega$.

For convenience, we assume Ω is a polygon in R^2, although this is not essential to our analysis. Furthermore, our main results remain valid for the case of Dirichlet boundary conditions on part or all of $\partial\Omega$.

A standard weak formulation of (1.1) is: Find $u \in \mathcal{H}^1(\Omega)$, such that

$$a(u, v) = (f, v), \quad \text{for all } v \in \mathcal{H}^1(\Omega), \tag{1.3}$$

where

$$a(u, v) = \int_\Omega a\nabla u \cdot \nabla v + b \cdot \nabla uv + cuv \, dx, \tag{1.4}$$

and $(\cdot, \cdot)$ is the usual $\mathcal{L}_2(\Omega)$ inner product. Note that in general $a(\cdot, \cdot)$ is nonsymmetric and possibly indefinite.

Let $\mathcal{H}^k(\Omega)$ denote the Sobolev space equipped with the norm

$$\|u\|_k^2 = \sum_{|\beta| < k} \|D^\beta u\|_{\mathcal{L}_2(\Omega)}^2$$

for k a nonnegative integer. For $k > 0$ and nonintegral, $\mathcal{H}^k$ will be defined by interpolation and for $k < 0$, $\mathcal{H}^k$ will be defined as the dual of $\mathcal{H}^{-k}$ [1], [12].

We assume a modest amount of elliptic regularity for the solution of (1.3). Specifically, we assume that (1.3) has a unique solution, and that there exists a fixed constant $0 < \alpha \leqq 1$, which does not depend on f, $(\alpha = \alpha(\Omega, a, b, c))$, such that $u \in \mathcal{H}^{1+\alpha}(\Omega)$ provided $f \in \mathcal{H}^{\alpha-1}(\Omega)$ (at least), and

$$\|u\|_s \leqq C(\Omega, a, b, c)\|f\|_{s-2}, \qquad 1 \leqq s \leqq 1 + \alpha. \tag{1.5}$$

We shall also have occasion to consider the formal adjoint problem

$$\begin{aligned} &\mathcal{L}^* u = f \quad \text{in } \Omega, \\ &a\frac{\partial u}{\partial n} + (b \cdot n)u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

$$\mathcal{L}^* u = -\nabla \cdot (a \nabla u) - \nabla \cdot (bu) + cu,$$

and the corresponding weak form: Find $u \in \mathcal{H}^1(\Omega)$ such that

$$a^*(u, v) \equiv a(v, u) = (f, v), \quad \text{for all } v \in \mathcal{H}^1(\Omega). \tag{1.6}$$

We assume (1.6) has a unique solution $u \in \mathcal{H}^{1+\alpha}(\Omega)$ for $f \in \mathcal{H}^{\alpha-1}(\Omega)$ and that (1.5) is valid in this case as well.

We now consider the finite element approximation of (1.3). Let $\mathcal{T}_1$ be a "coarse" triangulation of Ω. For $T \in \mathcal{T}_1$ define h_T to be the diameter of the circumscribing circle for T, d_T to be the diameter of the inscribing circle divided by h_T, and set

$$h_1 = \max_{T \in \mathcal{T}_1} h_T, \quad \delta_0 = \min_{T \in \mathcal{T}_1} d_T, \quad \delta_1 = \min_{T \in \mathcal{T}_1} h_T / h_1. \tag{1.7}$$

The constant δ_0 is a measure of the shape regularity of $\mathcal{T}_1$, and δ_1 a measure of its uniformity.

In our multilevel scheme, we will require a nested sequence of refined triangulations $\mathcal{T}_j$, $j \geqq 1$, generated recursively, beginning from $\mathcal{T}_1$, as follows: For each $T \in \mathcal{T}_j$ construct four triangles in $\mathcal{T}_{j+1}$ by pairwise connecting the midpoints of the edges, of T. From this definition, $\mathcal{T}_j$, $j > 1$, has shape regularity and uniformly constants δ_0 and δ_1 equal to those of $\mathcal{T}_1$, and has $h_j = \max_{T \in \mathcal{T}_j} h_T = h_1 2^{1-j}$.

We remark in passing that our analysis easily extends to other refinement schemes, (e.g., dividing $T \in \mathcal{T}_j$ into nine rather than four triangles in $\mathcal{T}_{j+1}$), to triangles with one curved (boundary) edge, and to some other classes of elements (e.g., rectangles). Such extensions are pursued in [3] for the self-adjoint case and will not be considered here.

Corresponding to each triangulation $\mathcal{T}_j$, we define an N_j-dimensional space $\mathcal{M}_j \subset \mathcal{H}^1(\Omega)$, of C^0 piecewise linear polynomials

$$\mathcal{M}_j = \{\varphi \mid \varphi \in C^0(\bar{\Omega}), \varphi \mid_{T \in \mathcal{T}_j} \text{is linear}\}.$$

Given our assumptions on $\mathcal{T}_j$, the spaces $\mathcal{M}_j$ can be shown to satisfy the following standard approximation property [12]: If $u \in \mathcal{H}^s(\Omega)$, $1 \leqq s \leqq 1 + \alpha$, then there exists $u_j \in \mathcal{M}_j$ such that

$$\|u - u_j\|_0 + h_j \|u - u_j\|_1 \leqq C(\delta_0, \delta_1, \Omega) h_j^s \|u\|_s. \tag{1.8}$$

Since the triangulations are nested we have $\mathcal{M}_j \subset \mathcal{M}_{j+1}$, $j \geqq 1$. Our analysis could be extended to spaces of higher degree polynomials, as in [3]; the essential properties required are (1.8) and the inclusion relation.

The finite element approximation of u in (1.3) is the function $u_j \in \mathcal{M}_j$ satisfying

$$a(u_j, v) = (f, v), \quad \text{for all } v \in \mathcal{M}_j. \tag{1.9}$$

Once a basis for $\mathcal{M}_j$ has been selected, (1.9) can be formulated as a set of linear equations of order N_j.

We assume (1.9) has a unique solution for $j \geqq 1$. (For h_1 sufficiently small, this can be proved using arguments similar to those in [1].) It follows that the adjoint problem

$$a^*(u_j, v) = (f, v), \quad \text{for all } v \in \mathcal{M}_j \tag{1.10}$$

also has a unique solution $u_j \in \mathcal{M}_j$ for $j \geqq 1$.

In the course of our investigation, it will be necessary to consider a generalization of (1.9) and (1.10): find $z \in \mathcal{M}_j$ such that

$$a(z, v) = G(v), \quad \text{for all } v \in \mathcal{M}_j, \tag{1.11}$$

or

$$a^*(z, v) = H(v), \quad \text{for all } v \in \mathcal{M}_j, \tag{1.12}$$

where $G(v)$ and $H(v)$ are linear functionals defined for $v \in \mathcal{M}_j$.

2. The multilevel methods. In this section we describe two multilevel iterative methods. Both schemes involve two recursive processes: (i) a single iteration of the j-level scheme for a problem of the form (1.11), and (ii) the overall multilevel procedure in which the problems (1.9) are solved sequentially for $j = 1, 2, \cdots$. The second of these processes is identical for both schemes; they differ only in the definition of the single j-level iteration. We begin by inductively defining a single iteration of the j-level scheme for both methods.

In both cases one "iteration" of the 1-level scheme consists of solving (1.11) exactly. If $j \geqq 1$, one iteration of either method takes initial guess $z_0 \in \mathcal{M}_j$ to a final guess $z_{m+1} \in \mathcal{M}_j$ as follows: For method 1 (the nonsymmetric scheme) z_l, $1 \leqq l \leqq m$ is defined by

$$(z_l - z_{l-1}, v) = \lambda_{N_j}^{-1}\{G(v) - a(z_{l-1}, v)\}, \qquad v \in \mathcal{M}_j. \tag{2.1}$$

For method 2 (the symmetric scheme) z_l, $1 \leqq l \leqq m$, and auxiliary functions $w_l \in \mathcal{M}_j$, $1 \leqq l \leqq m$ are defined by

$$\begin{aligned} (w_l, v) &= \tau_j^{-1}\{G(v) - a(z_{l-1}, v)\}, \qquad v \in \mathcal{M}_j, \\ (z_l - z_{l-1}, v) &= a^*(w_l, v), \qquad v \in \mathcal{M}_j. \end{aligned} \tag{2.2}$$

For both methods z_{m+1} is computed from z_m as follows: Let $q \in \mathcal{M}_{j-1}$ be the approximation of $\bar{q} \in \mathcal{M}_{j-1}$, generated by applying $p > 1$ iterations of the appropriate $(j-1)$-level scheme to the residual equations

$$a(\bar{q}, v) = G(v) - a(z_m, v) \equiv \bar{G}(v), \qquad v \in \mathcal{M}_{j-1} \tag{2.3}$$

starting from initial guess zero. Then set

$$z_{m+1} = z_m + q. \tag{2.4}$$

(Note that since $q \in \mathcal{M}_{j-1} \subset \mathcal{M}_j$, $z_{m+1} \in \mathcal{M}_j$.) Equations (2.1) and (2.2) are called "smoothing" iterations. Their purpose is to damp out components of the error which oscillate on the order of h_j. The parameters λ_{N_j} and τ_j are defined by

$$\lambda_{N_j} = \max_{v \in \mathcal{M}_j} \int_\Omega \frac{a|\nabla v|^2 + v^2\, dx}{\|v\|_0^2}, \tag{2.5}$$

$$\mu_{N_j}^2 = \max_{v \in \mathcal{M}_j} \frac{\|\varphi\|_0^2}{\|v\|_0^2}, \quad \text{where } a(v, \chi) = (\varphi, \chi) \text{ for all } \chi \in \mathcal{M}_j, \tag{2.6}$$

$$\tau_j = \max(\mu_{N_j}^2, \lambda_{N_j}^2). \tag{2.7}$$

Simple homogeneity arguments show that

$$\begin{aligned} \lambda_{N_j} &\leqq C(a, \Omega, \delta_0, \delta_1) h_j^{-2}, \\ \mu_{N_j}^2 &\leqq C(a, b, c, \Omega, \delta_0, \delta_1) h_j^{-4}. \end{aligned} \tag{2.8}$$

The constants λ_{N_j} and τ_j need not be calculated explicitly; replacing them by upper bounds of the form (2.8) is sufficient. Equation (2.1) requires the solution of a linear system involving the mass (Gram) matrix at each step. This was done for convenience. In practice, one usually computes with the standard nodal basis for $\mathcal{M}_j$. If M denotes the

mass matrix with respect to this basis and D denotes its diagonal, then there exists a constant $\mu = \mu(\Omega, \delta_0, \delta_1) > 0$, which is independent of j, such that

$$\mu^{-1} \leqq XMX/X^TDX \leqq \mu$$

for all $X \in R^{N_j}$ [12] (D itself will be comparable to a scalar multiple of the identity). In this case (2.1) becomes equivalent to an underrelaxed Jacobi-like iteration. A more complete theoretical discussion of this point can be found in [3, Cor. 2 of Thm. 1].

If the bilinear form $a(\cdot, \cdot)$ is indefinite, then application of (2.1) may cause smoother components of the error to grow; thus an overall increase in the error may be the net result of this step. (This is in contrast to the symmetric positive definite case, when the smoothing process always reduces the overall error, while being especially effective on the oscillations [3].) Thus, unlike the positive definite case, a proper choice of the integer parameter m is crucial to the success of the algorithm; choosing m too large can cause divergence of the method. Iterations like (2.1), (2.3)–(2.4) have been proposed and analyzed by several authors including Bakhvalov [2], Brandt [5], and Hackbusch [7].

It is the desire to have a smoothing iteration which is guaranteed to reduce the overall error that motivates (2.2). If we let $\hat{A}$ (M) denote the stiffness (mass) matrix and $Z_l \in R^{N_j}$ be the N_j vector corresponding to z_l, then (2.2) can be formulated in matrix terms as

$$M(Z_l - Z_{l-1}) = \tau_j^{-1} \hat{A}^T M^{-1} \{G - \hat{A} Z_{l-1}\}, \tag{2.9}$$

which is a symmetric iteration corresponding to a fourth order problem. Since $\hat{A}^T M^{-1} \hat{A}$ is symmetric and positive definite, this iteration will decrease the overall error. However, the condition number of $\hat{A}^T M^{-1} \hat{A}$ is essentially the square of the condition number of $\hat{A}$, so the rate of convergence can be slow; fortunately, this is not very serious, since we are primarily interested in the most oscillatory components of the error, for which the rate of convergence is still quite good. As in the case of (2.1), once we choose the nodal basis, (2.9) becomes equivalent to an underrelaxed Jacobi iteration for $\hat{A}^T \hat{A}$.

Once the error has been smoothed, using either (2.1) or (2.2), the two methods become identical. We expect the error now to be well approximated by a function in a space of smaller dimension, i.e., $\mathcal{M}_{j-1}$. Thus in (2.3) we compute an approximation of an elliptic projection of the error into $\mathcal{M}_{j-1}$, and in (2.4) we add it to z_m. The parameter p can be any integer greater than one; in practice, we usually take $p = 2$.

Let $\varepsilon_l = z - z_l \in \mathcal{M}_j$, $0 \leqq l \leqq m+1$ denote the error. Then we have the following convergence result for the nonsymmetric scheme.

THEOREM 1. *Let z_l, $1 \leqq l \leqq m+1$, be defined using* (2.1), (2.3) *and* (2.4). *Let $p > 1$ be any integer, and assume* (1.5) *and* (1.8) *hold. Then there exists a constant $0 \leqq \gamma < 1$ and an integer $m \geqq 1$, depending on $(\bar{a}, \underline{a}, \bar{b}, \Omega, \delta_0, \delta_1, h_1)$ but not on h_j or j, such that, if h_1 is sufficiently small and*

$$\|\bar{q} - q\|_1 \leqq \gamma^p \|\bar{q}\|_1, \tag{2.10}$$

then

$$\|\varepsilon_{m+1}\|_1 \leqq \gamma \|\varepsilon_m\|_1. \tag{2.11}$$

Theorem 1 will be proved in § 3. Similar results can be found in [2], [7], [9]. Inequality (2.10) is an induction hypothesis asserting that the $(j-1)$-level scheme can reduce the error by a factor of at least γ in each iteration. Also in § 3 we will prove the following theorem.

THEOREM 2. *Let z_l, $1 \leqq l \leqq m+1$ be defined by* (2.2), (2.3) *and* (2.4). *Then if the hypotheses of Theorem* 1 *hold, the conclusion remains valid.*

Before presenting the overall multilevel process, let us consider the work estimate for one iteration of the j-level scheme. Let $F(N)$ denote a bound on the cost of reducing the error by a factor of γ for a problem of N unknowns. Clearly F is nondecreasing and $F(N_1) = F_1$ is the cost of solving a level-1 problem. If any reasonable method is used for the smoothing iteration, then (2.1) or (2.2) have cost bound of the form $s_1 N_j m$ where s_1 is a constant independent of j. The cost of computing the residual equations (right-hand side) in (2.3) and z_{m+1} in (2.4) is bounded by $s_2 N_j$ where s_2 is independent of j. the cost of p iterations of the $(j-1)$-level scheme is bounded by $pF(N_{j-1})$. Thus, since $N_{j-1} \sim N_j/4$,

$$F(N_j) \sim pF\left(\frac{N_j}{4}\right) + (s_1 m + s_2)N_j. \tag{2.12}$$

Solution of the difference equation (2.12) leads to [10],

$$F(N) \sim \begin{cases} F_1\left(\dfrac{N_j}{N_1}\right)^{\log p/\log 4} + \dfrac{(s_1 m + s_2)}{(1-p/4)} N_1 \left(\left(\dfrac{N_j}{N_1}\right) - \left(\dfrac{N_j}{N_1}\right)^{\log p/\log 4}\right), & p \neq 4, \\ F_1\left(\dfrac{N_j}{N_1}\right) + \dfrac{(s_1 m + s_2)}{\log 4} N_j \log\left(\dfrac{N_j}{N_1}\right), & p = 4. \end{cases} \tag{2.13}$$

Clearly for $p = 2, 3$ the bound is asymptotically (as $j \to \infty$) of order N_j. Hereafter, we shall assume $p = 2$; then the cost of reducing the error by a fixed amount γ is proportional to the number of unknowns.

The recursion involved in the overall multilevel scheme eliminates the $\log N_j$ factor in the work estimate associated with producing an $O(h_j^\alpha)$ solution to the discrete equations. On the basis of the regularity assumption (1.5) and the approximation property (1.8), it can be proved [11] that for h_1 sufficiently small the solutions u_j of (1.9) satisfy

$$\|u - u_j\|_1 \leqq \mathcal{K} h_j^\alpha, \qquad j \geqq 1, \tag{2.14}$$

where $\mathcal{K} = \mathcal{K}(u, a, b, c, \Omega, \delta_0, \delta_1)$. Let $\tilde{u}_j$, $j \geqq 1$ be approximate solutions of (1.9) generated as follows: For $j = 1$, compute $\tilde{u}_j$ by solving (1.9) exactly. For $j > 1$, apply r iterations of the j-level scheme (either symmetric or nonsymmetric version) to (1.9) starting from initial guess $\tilde{u}_{j-1}$ and obtaining final guess $\tilde{u}_j$. If $e_j = \|u_j - \tilde{u}_j\|_1$, then Theorem 1 or 2 implies

$$e_j \leqq \gamma^r \|u_j - \tilde{u}_{j-1}\| \leqq \gamma^r \{\|u_j - u\| + \|u_{j-1} - u\| + e_{j-1}\} \leqq \gamma^r \{\mathcal{K}(1 + 2^\alpha) h_j^\alpha + e_{j-1}\},$$

where we have used (2.14) and $h_{j-1} = 2h_j$. The solution of the majorizing difference equation yields

$$e_j \leqq \mathcal{K} \delta h_j^\alpha, \qquad j \geqq 1, \tag{2.15}$$

$$\delta = \gamma^r \frac{(1 + 2^\alpha)}{(1 - \gamma^r 2^\alpha)},$$

where we assume (2.15) for $j = 1$ and that $\gamma^r 2^\alpha < 1$. Clearly the choice of r is independent of j. Thus from (2.15)

$$\|u - \tilde{u}_j\|_1 \leqq \|u - u_j\|_1 + e_j \leqq (1 + \delta)\mathcal{K} h_j^\alpha, \qquad j \geqq 1. \tag{2.16}$$

The cost of computing $\tilde{u}_j$ is bounded by (using (2.13) for $p=2$)

$$(2.17)\quad F(N_1)+r\sum_{k=2}^{j}F(N_k)\sim\left\{1+2r\left(\frac{N_j}{N_1}\right)^{1/2}\right\}F_1+\frac{8}{3}N_jr(s_1m+s_2)\left\{1-\frac{3}{2}\left(\frac{N_1}{N_j}\right)^{1/2}\right\}.$$

Asymptotically this is of order N_j. Thus we have

THEOREM 3. *Let the hypotheses of Theorem* 1 *hold. Then the cost of producing a solution* $\tilde{u}_j$ *satisfying* (2.16) *using a multilevel scheme employing either symmetric or nonsymmetric smoothing iterations is proportional to* N_j *as* $j\to\infty$.

Having selected $p=2$ on the basis of (2.13), we now consider qualitatively the selection of the parameters m and r. To obtain the estimate (2.16) we must have

$$(2.18)\qquad (\gamma(m))^r=\nu<2^{-\alpha}.$$

Using (2.18) to eliminate r in the estimate (2.17) yields the cost as a function of m only:

$$(2.19)\quad \text{cost}\sim\left(1+2\frac{\log\nu}{\log\gamma}\left(\frac{N_j}{N_1}\right)^{1/2}\right)F_1+\frac{8}{3}N_j\frac{\log\nu}{\log\gamma}(s_1m+s_2)\left(1-\frac{3}{2}\left(\frac{N_1}{N_j}\right)^{1/2}\right).$$

The minimum cost with respect to choice of m is achieved when

$$(2.20)\qquad \frac{1}{\mathcal{S}}=\left\{\frac{\gamma\log\gamma}{d\gamma/dm}-m\right\},\qquad \mathcal{S}=\frac{s_1}{\frac{F_1}{2N_1}\left\{\frac{2}{3}\left(\frac{N_j}{N_1}\right)^{1/2}-1\right\}+s_2}.$$

Note $\mathcal{S}$ is essentially independent of the partial differential equation, and depends primarily on the details of the multilevel iteration; as $j\to\infty$, $\mathcal{S}\to s_1/s_2$ and measures the cost of a single smoothing iteration relative to the cost of computing the residual in (2.3) and computing z_{m+1} in (2.4). For the symmetric positive definite case ($b_i=0$, $c>0$), we have shown that $\gamma\leqq D(a,c,\Omega,\delta_0,\delta_1)m^{-\alpha/2}$. We believe this estimate to be sharp. Thus, if we assume $\gamma\sim Dm^{-\alpha/2}$ for $D<1$ (so that $\gamma<1$ for $m=1$, consistent with the empirically observed behavior of our multilevel scheme), (2.20) becomes

$$(2.21)\qquad \frac{1}{m}=\mathcal{S}(\log m-1+\mathcal{P}),\qquad \mathcal{P}=\frac{-2\log D}{\alpha}\geqq 0.$$

The closest integer solutions of (2.21) for typical values of $\mathcal{S}$ and $\mathcal{P}$ are summarized in Table 1.

TABLE 1
Optimal value of m

$\mathcal{P}$	$\mathcal{S}$				
	4	2	1	1/2	1/4
0	2-3	3-4	3-4	4-5	5-6
1/4	2-3	2-3	2-3	3-4	3-4
1/2	1-2	2-3	2-3	3-4	3-4

Note that the solution of (2.21) does not depend strongly on $\mathcal{P}(\alpha,a,c,\Omega,\delta_0,\delta_1)$, i.e., the details of the differential equation and the initial triangulation.

Also, when the cost of smoothing is comparatively large (large $\mathcal{S}$), m is small. When the costs of computing the residual and solving (1.12) for $j=1$ are comparatively large, the optimal value of m increases.

When we treat the nonsymmetric problem using the symmetric smoothing scheme (2.2) we expect that asymptotically, for h_1 sufficiently small, $\gamma \sim Dm^{-\alpha/4}$, where the factor of 4 is the effect of squaring the condition number. Behavior of this form is similar to the symmetric positive definite case, and the results in Table 1 are applicable.

On the other hand, when the nonsymmetric smoothing scheme (2.1) is employed, we expect γ to behave more like $\gamma \sim D(1+Ch_1)^m m^{-\alpha/2}$, where the factor of $(1+Ch_1)^m$ is due to growth of the smoother components of the error during the smoothing phase.

The condition $\gamma < 1$ necessarily implies

$$m \leqq \frac{(\alpha/2 \log m - \log D)}{\log (1+Ch_1)}. \tag{2.22}$$

By picking h_1 sufficiently small, we can make the right-hand side of (2.22) sufficiently large that (2.22) will be satisfied for at least some values of m. Nonetheless, (2.22) implies that, unlike the symmetric iteration (2.2), for a given problem and initial mesh $\mathcal{T}_1$, there is an integer $\bar{m}$ such that the nonsymmetric iteration may diverge for $m \geqq \bar{m}$. We will see examples of this in § 4.

3. Proofs of Theorems 1 and 2. In the course of proving Theorems 1 and 2 it will be convenient to consider three sets of orthogonal basis functions for the space $\mathcal{M}_j$ (note (2.1)–(2.4) do not depend on the choice of basis).

Let $a(u, v) = a_1(u, v) + b(u, v)$, where

$$\begin{aligned} a_1(u, v) &= \int_\Omega a\nabla u \nabla v + uv \, dx, \\ b(u, v) &= \int_\Omega b\nabla uv + (c-1)uv \, dx. \end{aligned} \tag{3.1}$$

Note that $a_1(\cdot, \cdot)$ is symmetric and positive definite. Define eigenvalues λ_k and corresponding eigenfunctions $\psi_k \in \mathcal{M}_j$ by

$$\begin{aligned} a_1(\psi_k, v) &= \lambda_k(\psi_k, v), \quad \text{for all } v \in \mathcal{M}_j, \\ \|\psi_k\|_0 &= 1. \end{aligned} \tag{3.2}$$

We order the eigenvalues such that $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{N_j}$. Then λ_{N_j} is also given by (2.5). Note that we may require

$$(\psi_i, \psi_k) = \delta_{ik}, \qquad a_1(\psi_i, \psi_k) = \lambda_k \delta_{ik}. \tag{3.3}$$

Let $\varphi_k, \theta_k \in \mathcal{M}_j$ and μ_k, $0 \leqq k \leqq N_j$, satisfy

$$\begin{aligned} a(\varphi_k, v) &= \mu_k(\theta_k, v), \quad \text{for all } v \in \mathcal{M}_j, \\ a^*(\theta_k, v) &= \mu_k(\varphi_k, v), \quad \text{for all } v \in \mathcal{M}_j, \\ \|\theta_k\|_0 &= \|\varphi_k\|_0 = 1. \end{aligned} \tag{3.4}$$

We order the singular values such that $0 < \mu_1^2 \leqq \mu_2^2 \leqq \cdots \leqq \mu_{N_j}^2$. Then $\mu_{N_j}^2$ is also given by (2.6). Again, we may require that

$$(\theta_i, \theta_k) = \delta_{ik}, \qquad (\varphi_i, \varphi_k) = \delta_{ik}. \tag{3.5}$$

The sets $\{\psi_i\}$, $\{\varphi_i\}$ and $\{\theta_i\}$ are all orthogonal bases for $\mathcal{M}_j$. If we let $\hat{A}$, $\hat{A}_1$, $\hat{B}$, M denote the matrices corresponding to $a(\cdot, \cdot)$, $a_1(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $(\cdot, \cdot)$ with respect to an arbitrary basis, and let $\hat{\Psi}_k$, $\hat{\Phi}_k$ and $\hat{\Theta}_k \in R^{N_j}$ correspond to ψ_k, φ_k and θ_k represented in

that basis, then the matrix analogues of (3.2) and (3.4) are

$$\hat{A}_1\hat{\Psi}_k = \lambda_k M\hat{\Psi}_k, \quad \hat{A}M^{-1}\hat{A}^T\hat{\Theta}_k = \mu_k^2 M\hat{\Theta}_k, \quad \hat{A}^T M^{-1}\hat{A}\hat{\Phi}_k = \mu_k^2 M\hat{\Phi}_k.$$

Since M is symmetric and positive definite for an arbitrary choice of basis for $\mathcal{M}_j$, we will usually employ the scaled matrices $A = M^{-1/2}\hat{A}M^{-1/2}$, $A_1 = M^{-1/2}\hat{A}_1M^{-1/2}$, and $B = M^{-1/2}\hat{B}M^{-1/2}$, and the corresponding vectors $\Psi_k = M^{1/2}\hat{\Psi}_k$, $\Phi_k = M^{1/2}\hat{\Phi}_k$, and $\Theta_k = M^{1/2}\hat{\Theta}_k$. Then

$$A_1\Psi_k = \lambda_k\Psi_k, \quad AA^T\Theta_k = \mu_k^2\Theta_k, \quad A^TA\Phi_k = \mu_k^2\Phi_k.$$

If we use the basis $\{\psi_k\}$ for $\mathcal{M}_j$, then $M = I$ and A_1 is diagonal; if we use $\{\theta_k\}$ or $\{\varphi_k\}$ then $M = I$. If $\chi \in \mathcal{M}_j$ then $\chi = \sum_{k=1}^{N_j} c_k\psi_k = \sum_{k=1}^{N_j} c_k'\varphi_k = \sum_{k=1}^{N_j} c_k''\theta_k$. Define discrete norms $|||\chi|||_{s,\nu}$, $-2 \leqq s \leqq 2$, $\nu = \theta, \varphi, \psi$, by

$$\begin{aligned} |||\chi|||_{s,\psi}^2 &= \sum_{k=1}^{N_j} c_k^2\lambda_k^s, \\ |||\chi|||_{s,\varphi}^2 &= \sum_{k=1}^{N_j} (c_k')^2\mu_k^s, \\ |||\chi|||_{s,\theta}^2 &= \sum_{k=1}^{N_j} (c_k'')^2\mu_k^s. \end{aligned} \tag{3.6}$$

It is convenient to associate with every $\chi \in \mathcal{M}_j$ a vector $X \in R^{N_j}$ consistent with the definitions of A, A_1 and B above. Let $\{\rho_k\}$ be a basis for $\mathcal{M}_j$. then $\chi = \sum_{k=1}^{N_j} \hat{X}_k\rho_k$ induces a natural correspondence between χ and $\hat{X} \in R^{N_j}$. We shall associate χ instead with the scaled vector $X = M^{1/2}\hat{X}$. Then, for $-2 \leqq s \leqq 2$,

$$\begin{aligned} \|A_1^{s/2}X\|_{l_2}^2 &\equiv X^TA_1^sX = |||\chi|||_{s,\psi}^2, \\ \|(A^TA)^{s/4}X\|_{l_2}^2 &\equiv X^T(A^TA)^{s/2}X = |||\chi|||_{s,\varphi}^2, \\ \|(AA^T)^{s/4}X\|_{l_2}^2 &\equiv X^T(AA^T)^{s/2}X = |||\chi|||_{s,\theta}^2. \end{aligned}$$

This is true for any choice of basis in $\mathcal{M}_j$. Note that $|||\chi|||_{0,\nu} \equiv \|\chi\|_0$, and we expect $|||\chi|||_{s,\nu}$ to correspond in some sense to $\|\chi\|_s$. This is made precise in Lemma 1.

LEMMA 1. *Let $\chi \in \mathcal{M}_j$ and let* (1.5) *and* (1.8) *hold. Then, for $\nu = \psi$, φ and θ,*

$$C^{-1}\|\chi\|_s \leqq |||\chi|||_{s,\nu} \leqq C\|\chi\|_s, \qquad 0 \leqq s \leqq 1, \tag{3.7}$$

$$|||\chi|||_{-(1+\alpha),\nu} \leqq C\{\|\chi\|_{-(1+\alpha)} + h_j^{2\alpha}\|\chi\|_{\alpha-1}\}; \tag{3.8}$$

for $\nu = \psi$, $C = C(a, \Omega, \delta_0, \delta_1)$; for $\nu = \theta, \varphi$, $C = C(a, b, c, \Omega, \delta_0, \delta_1, h_1)$

The proof of Lemma 1 is long and technical and can be found in the Appendix. the argument used for (3.7) is a variation of an argument used by Bank and Dupont in [3], while that used in (3.8) is a variation of an argument used by Thomeé [13].

LEMMA 2. *Let $\chi \in \mathcal{M}_j$ satisfy*

$$(\chi, v) = 0, \quad \text{for all } v \in \mathcal{M}_{j-1},$$

and let the hypotheses of Lemma 1 hold; then for $\nu = \psi, \varphi, \theta$

$$|||\chi|||_{-(1+\alpha),\nu} \leqq Ch_j^{1+\alpha}\|\chi\|_0, \tag{3.9}$$

for $C = C(a, \Omega, \delta_0, \delta_1)$ for $\nu = \psi$ and $C = C(a, b, c, \Omega, \delta_0, \delta_1, h_1)$ for $\nu = \varphi, \theta$.

Proof. Let $1-\alpha \leqq s \leqq 1+\alpha$ and let $\eta \in \mathcal{M}_{j-1}$; then

$$\|\chi\|_{-s} = \sup_{v \in \mathcal{H}^s} \frac{(\chi, v)}{\|v\|_s} = \sup_{v \in \mathcal{H}^s} \frac{(\chi, v-\eta)}{\|v\|_s} \leqq C h_j^s \|\chi\|_0, \tag{3.10}$$

where we have used the fact that $h_{j-1} = 2h_j$. This result for $s = 1 \pm \alpha$, together with (3.8), gives

$$|||\chi|||_{-(1+\alpha),\nu} \leqq C\{\|\chi\|_{-(1+\alpha)} + h_j^{2\alpha}\|\chi\|_{\alpha-1}\} \leqq C h_j^{1+\alpha}\|\chi\|_0.$$

Let $\varepsilon_l = z - z_l$, $0 \leqq l \leqq m$ define the error. Then we have the following lemma.

LEMMA 3. *Let z_l, $1 \leqq l \leqq m$, be defined using the nonsymmetric scheme* (2.1). *Then, if h_1 is sufficiently small,*

$$|||\varepsilon_m|||_{1,\psi} \leqq C(1+Ch_1)^m \|\varepsilon_0\|_1, \tag{3.11}$$

$$|||\varepsilon_m|||_{1+\alpha,\psi} \leqq C h_j^{-\alpha} m^{-\alpha/2} (1+Ch_1)^m \|\varepsilon_0\|_1, \tag{3.12}$$

where $C = C(a, b, c, \Omega, \delta_0, \delta_1, h_1)$.

Proof. Let A, A_1, B, M be the matrices defined above and let $E_l \in R^{N_j}$ correspond to $\varepsilon_l \in \mathcal{M}_j$. Then, from (2.1), we have

$$E_m = (I - \lambda_{N_j}^{-1} A) E_{m-1} = (I - \lambda_{N_j}^{-1} A)^m E_0. \tag{3.13}$$

Noting that A_1 is symmetric and positive definite, and thus has a positive square root, we let $M_1 = I - \lambda_{N_j}^{-1} A_1$, $B_1 = A_1^{1/2} B A_1^{-1/2}$. Since $A = A_1 + B$ we can manipulate (3.13) into the form

$$A_1^{(1+s)/2} E_m = A_1^{s/2} (M_1 - \lambda_{N_j}^{-1} B_1)^m A_1^{1/2} E_0. \tag{3.14}$$

We now verify the following estimates:

$$\|A_1^{s/2} M_1^t\|_{l_2} \leqq C t^{-s/2} h_j^{-s}, \qquad t \geqq 1, \qquad 0 \leqq s \leqq \alpha, \tag{3.15}$$

$$\|B_1\|_{l_2} \leqq C h_j^{-1}. \tag{3.16}$$

Since (3.14) is independent of the choice of basis in $\mathcal{M}_j$, we will select the basis $\{\psi_k\}$; then A_1 and M_1 are diagonal, and we have

$$\|A_1^{s/2} M_1^t\|_{l_2} = \sup_k \lambda_k^{s/2} \left(1 - \frac{\lambda_k}{\lambda_{N_j}}\right)^t \leqq \lambda_{N_j}^{s/2} \sup_{x \in [0,1]} x^{s/2}(1-x)^t \leqq C \lambda_{N_j}^{s/2} t^{-s/2}.$$

Since $\lambda_{N_j}^{s/2} \leqq C h_j^{-s}$ by (2.8), (3.15) has been verified. To show (3.16), again choose $\{\psi_k\}$ as the basis for $\mathcal{M}_j$, and let $\chi \in \mathcal{M}_j$ correspond to $X \in R^{N_j}$. Then

$$\|B_1\|_{l_2}^2 = \sup_{\substack{Y \in R^{N_j} \\ Y \neq 0}} \frac{Y^T A_1^{-1/2} B^T A_1 B A_1^{-1/2} Y}{Y^T Y} = \sup_{\substack{X \in R^{N_j} \\ X \neq 0}} \frac{X^T B^T A_1 B X}{X^T A_1 X}$$

$$= \sup_{\chi \in \mathcal{M}_j} \sum_{i=1}^{N_j} \frac{\lambda_i (b\nabla\chi + (c-1)\chi, \psi_i)^2}{|||\chi|||_{1,\psi}^2}.$$

But the $(b\nabla\chi + (c-1)\chi, \psi_i)$ are just the coefficients of the $\mathcal{L}_2(\Omega)$-projection of $b\nabla\chi + (c-1)\chi$ into $\mathcal{M}_j$. Thus

$$\|B_1\|_{l_2}^2 \leqq \lambda_{N_j} \sup_{\chi \in \mathcal{M}_j} \sum_{i=1}^{N_j} \frac{(b\nabla\chi + (c-1)\chi, \psi_i)^2}{|||\chi|||_{1,\psi}^2}$$

$$\leqq \lambda_{N_j} \frac{\|b\nabla\chi + (c-1)\chi\|_0^2}{|||\chi|||_{1,\psi}^2} \leqq C \lambda_{N_j} \leqq C h_j^{-2}.$$

Taking square roots then verifies (3.16).

To prove (3.11), use (3.14) and (3.16) to see

$$\begin{aligned}|||\varepsilon_m|||_{1,\psi} &\leqq (\|M_1\|_{l_2} + \lambda_{N_j}^{-1}\|B_1\|_{l_2})^m |||\varepsilon_0|||_{1,\psi} \\ &\leqq (1+Ch_j)^m |||\varepsilon_0|||_{1,\psi} \leqq C(1+Ch_1)^m \|\varepsilon_0\|_1,\end{aligned}$$

where we have used $h_j \leqq h_1$, Lemma 1 and $\|M_1\|_{l_2} \leqq 1$.

To prove (3.12), expand (3.14) in terms grouped according to the multiplicity of B_1. The term with no B_1 is $\|A_1^{\alpha/2} M_1^m\|_{l_2} \leqq Cm^{-\alpha/2} h_j^{-j}$, where we have used (3.15) with $s = \alpha$, $t = m$. Terms with one occurrence of B_1 are

$$\begin{aligned}\left\| \lambda_{N_j}^{-1} \sum_{k=1}^{m} A_1^{\alpha/2} M_1^{k-1} B_1 M_1^{m-k} \right\|_{l_2} &\leqq Ch_j^2 \sum_{k=1}^{m} \|A_1^{\alpha/2} M_1^{k-1}\| \|B_1\| \|M_1^{m-k}\| \\ &\leqq Ch_j^{1-\alpha} \sum_{k=1}^{m} k^{-\alpha/2} \leqq Ch_j^{1-\alpha} m^{1-\alpha/2},\end{aligned}$$

where we have used (3.15) with $s = \alpha$, and (3.16). The remaining terms are bounded in similar fashion. Thus, from (3.14),

$$\begin{aligned}|||\varepsilon_m|||_{1+\alpha,\psi} &\leqq Ch_j^{-\alpha} m^{-\alpha/2}(1 + Ch_1 m + \cdots + (Ch_1)^m) |||\varepsilon_0|||_{1,\psi} \\ &\leqq Ch_j^{-\alpha} m^{-\alpha/2}(1+Ch_1)^m \|\varepsilon_0\|_1,\end{aligned}$$

where we have used (3.7) for $s = 1$, $\nu = \psi$.

LEMMA 4. *Let z_l, $1 \leqq l \leqq m$ be defined using the symmetric scheme* (2.2). *Then*

$$|||\varepsilon_m|||_{1,\varphi} \leqq C\|\varepsilon_0\|_1, \tag{3.17}$$

$$|||\varepsilon_m|||_{1+\alpha,\varphi} \leqq Ch_j^{-\alpha} m^{-\alpha/4} \|\varepsilon_0\|_1, \tag{3.18}$$

where $C = C(a, b, c, \delta_0, \delta_1, h_1)$.

Proof. Define A, A_1, B, M and E_l as above and let $A_2 = A^T A$. Then (2.2) can be written in matrix form as

$$E_m = (I - \tau_j^{-1} A_2) E_{m-1} = (I - \tau_j^{-1} A_2)^m E_0. \tag{3.19}$$

Since A_2 is symmetric and positive definite,

$$A_2^{(1+s)/4} E_m = A_2^{s/4} \{I - \tau_j^{-1} A_2\}^m A_2^{1/4} E_0. \tag{3.20}$$

Using an argument similar to that used to prove (3.15), (choose the basis $\{\varphi_i\}$) we have

$$\|A_2^{s/4}(I - \tau_j^{-1} A_2)^m\|_{l_2} \leqq Ch_j^{-s} m^{-s/4}. \tag{3.21}$$

Estimates (3.17) and (3.18) are then immediate consequences of (3.19)–(3.21) with $s = 0$ and $s = \alpha$, and (3.7) with $\nu = \varphi$.

LEMMA 5. *Let z_m be defined using either* (2.1) *or* (2.2) *and let $\bar{q}$ be defined by* (2.3). *Then, for h_1 sufficiently small, and any $\chi \in \mathcal{M}_{j-1}$,*

$$\|\bar{q} - \varepsilon_m\|_1 \leqq Ch_j^{-1} \|\chi - \varepsilon_m\|_0, \tag{3.22}$$

$$\|\bar{q}\|_1 \leqq C\|\varepsilon_m\|_1, \tag{3.23}$$

where $C = C(a, b, c, \Omega, h_1, \alpha, \delta_0, \delta_1)$.

Proof. Our proof is based on the ideas of Schatz in [11]. First note, from (2.3), $a(\bar{q} - \varepsilon_m, v) = 0$ for all $v \in \mathcal{M}_{j-1}$. Let $\eta \in \mathcal{H}^1(\Omega)$ and $\zeta \in \mathcal{L}_2(\Omega)$ satisfy

$$a(\eta, v) = (\zeta, v), \quad \text{for all } v \in \mathcal{H}^1(\Omega).$$

Taking $v = \bar{q} - \varepsilon_m$, we have, for any $\chi \in \mathcal{M}_{j-1}$,

$$(\zeta, \bar{q} - \varepsilon_m) = a(\eta - \chi, \bar{q} - \varepsilon_m) \leqq C\|\eta - \chi\|_1 \|\bar{q} - \varepsilon_m\|_1$$
$$\leqq Ch_j^{\alpha}\|\eta\|_{1+\alpha}\|\bar{q} - \varepsilon_m\|_1 \leqq Ch_j^{\alpha}\|\zeta\|_{\alpha-1}\|\bar{q} - \varepsilon_m\|_1,$$

where we have used (1.5) and (1.8). Thus

$$\|\bar{q} - \varepsilon_m\|_{1-\alpha} \leqq Ch_j^{\alpha}\|\bar{q} - \varepsilon_m\|_1 \tag{3.24}$$

and

$$\|\bar{q} - \varepsilon_m\|_0 \leqq \|\bar{q} - \varepsilon_m\|_{1-\alpha} \leqq Ch_j^{\alpha}\|\bar{q} - \varepsilon_m\|_1. \tag{3.25}$$

Now, from (3.25),

$$a(\bar{q} - \varepsilon_m, \bar{q} - \varepsilon_m) \geqq C_1\|\bar{q} - \varepsilon_m\|_1^2 - C_2\|\bar{q} - \varepsilon_m\|_0^2 \geqq (C_1 - C_3h_j^{2\alpha})\|\bar{q} - \varepsilon_m\|_1^2.$$

Now $C_1 - C_3h_j^{2\alpha} \geqq C > 0$ for h_1 sufficiently small; thus

$$\|\bar{q} - \varepsilon_m\|_1^2 \leqq Ca(\bar{q} - \varepsilon_m, \bar{q} - \varepsilon_m) = Ca(\bar{q} - \varepsilon_m, \chi - \varepsilon_m) \leqq C\|\bar{q} - \varepsilon_m\|_1 \|\chi - \varepsilon_m\|_1, \tag{3.26}$$

for any $\chi \in \mathcal{M}_{j-1}$. To prove (3.23) use $\chi = 0$ to show

$$\|\bar{q}\|_1 \leqq \|\varepsilon_m\|_1 + \|\bar{q} - \varepsilon_m\|_1 \leqq C\|\varepsilon_m\|_1.$$

To prove (3.22) use $\|\chi - \varepsilon_m\|_1 \leqq Ch_j^{-1}\|\chi - \varepsilon_m\|_0$.

Proof of Theorem 1. First note $\varepsilon_{m+1} = \varepsilon_m - q$; thus, from Lemma 5 and the induction hypothesis (2.10),

$$\|\varepsilon_{m+1}\|_1 = \|\varepsilon_m - q\|_1 \leqq \|q - \bar{q}\|_1 + \|\varepsilon_m - \bar{q}\|_1 \leqq \gamma^p\|\bar{q}\|_1 + Ch_j^{-1}\|\varepsilon_m - \chi\|_0. \tag{3.27}$$

To bound the first term, use Lemmas 3 and 5 to get

$$\|\bar{q}\|_1 \leqq C\|\varepsilon_m\|_1 \leqq C|||\varepsilon_m|||_{1,\psi} \leqq C(1 + Ch_1)^m\|\varepsilon_0\|_1. \tag{3.28}$$

To bound the second term, let χ be the $\mathcal{L}_2(\Omega)$ projection of ε_m into $\mathcal{M}_{j-1}$. Then

$$\|\varepsilon_m - \chi\|_0^2 = (\varepsilon_m - \chi, \varepsilon_m) \leqq |||\varepsilon_m - \chi|||_{-(1+\alpha),\psi}|||\varepsilon_m|||_{1+\alpha,\psi}$$
$$\leqq Ch_jm^{-\alpha/2}(1 + Ch_1)^m\|\varepsilon_m - \chi\|_0\|\varepsilon_0\|_1, \tag{3.29}$$

where we have used Lemma 2 (3.9) and Lemma 3 (3.12). Combining (3.29)–(3.31) we have

$$\|\varepsilon_{m+1}\|_1 \leqq (C\gamma^p(1 + Ch_1)^m + Cm^{-\alpha/2}(1 + Ch_1)^m)\|\varepsilon_0\|_1. \tag{3.30}$$

To complete the proof of Theorem 1, note $p > 1$. Let $0 \leqq \gamma < 1$ be any number satisfying $C\gamma^{p-1/2} < \gamma/2$; pick m sufficiently large that $Cm^{-\alpha/2} < \gamma^{3/2}/2$ and pick h_1 sufficiently small that

$$(1 + Ch_1)^m \leqq \gamma^{-1/2}.$$

For such choices of γ, m and h_1, we have

$$\|\varepsilon_{m+1}\|_1 \leqq \gamma\|\varepsilon_0\|_1,$$

completing the proof of Theorem 1.

Proof of Theorem 2. Note that (3.27) remains valid for the symmetric scheme. To bound the first term, use Lemmas 4 and 5 to get

$$\|q\|_1 \leqq C\|\varepsilon_m\|_1 \leqq C\|\varepsilon_0\|_1. \tag{3.31}$$

To bound the second term, argue analogously to (3.29), using Lemma 4 (3.18), in place

of Lemma 3 to get

$$\|\varepsilon_m - \chi\|_0 \leqq Ch_j m^{-\alpha/4}\|\varepsilon_0\|_1. \tag{3.32}$$

Combining (3.27) with (3.31)–(3.32) yields

$$\|\varepsilon_{m+1}\|_1 \leqq C(\gamma^p + m^{-\alpha/4})\|\varepsilon_0\|_1. \tag{3.33}$$

Now let $0 \leqq \gamma < 1$ be such that $C\gamma^p < \gamma/2$, and choose m sufficiently large that $Cm^{-\alpha/4} < \gamma/2$.

4. Numerical illustrations. In this section we present several numerical examples intended to contrast the behavior of the symmetric and nonsymmetric schemes. These experiments were carried out using subroutine PLTMG, a Fortran program of Bank and Sherman. A discussion of the details of this program can be found in [4]. All experiments were done on a CDC6600 computer using single precision arithmetic and the MNF compiler.

We first consider the self-adjoint problem

$$\begin{aligned} -\Delta u + \lambda u &= f \quad \text{in } \Omega = (0,1)\times(0,1), \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

with true solution $u = \sin(\pi x)\sin(\pi y)$. The operator is positive definite for $\lambda > -2\pi^2$.

A uniform triangulation consisting of 9 vertices and 8 right triangles, as illustrated in Fig. 1, was provided to PLTMG. We shall refer to this triangulation as a 3×3 grid.

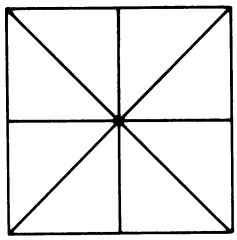

FIG. 1

Three additional meshes were generated by uniform refinement as described in § 1, resulting in 5×5, 9×9 and 17×17 meshes.

We solved (4.1) for various values of $\lambda \in [0, -36]$ using both symmetric and nonsymmetric schemes, with $\mathcal{T}_1$ the 3×3 mesh (a 4-level scheme), $\mathcal{T}_1$ the 5×5 mesh (a 3-level scheme), and $\mathcal{T}_1$ the 9×9 mesh (a 2-level scheme).

We measured two types of error. First, we measured the error of the exact finite element solution on the 17×17 grid, u_h, relative to the continuous solution u. Second, we measured the error in the computed multigrid solution on the 17×17 grid, $\tilde{u}_h$, relative to u_h. In the former case we define

$$\text{digits} = -\log(\|u_h - u\|_1/\|u\|_1) \tag{4.2}$$

and in the latter, we define

$$\text{digits} = -\log(\|\tilde{u}_h - u_h\|_1/\|u_h\|_1). \tag{4.3}$$

In Tables 2–5 we record the number of digits in the computed solution $\tilde{u}_h$, measured using (4.3), for $\lambda = -36$, for both 3- and 4-level schemes, and both the

TABLE 2
The 4-level nonsymmetric scheme for (4.1) *with* $\lambda = -36$

r/m	1	2	3	4
1	1.6	2.0	1.3	−.9
2	2.2	2.9	3.2	−1.5
3	2.7	3.7	3.5	−2.4
4	3.2	4.4	4.3	−3.1

(correct digits measured using (4.3))

TABLE 3
The 3-level nonsymmetric scheme for (4.1) *with* $\lambda = -36$

r/m	1	2	3	4
1	1.6	2.1	2.4	2.6
2	2.2	2.9	3.4	3.7
3	2.7	3.7	4.3	4.9
4	3.2	4.4	5.2	5.9

(correct digits measured using (4.3))

TABLE 4
The 4-level symmetric scheme for (4.1) *with* $\lambda = -36$

r/m	1	2	3	4
1	.9	1.3	1.5	1.7
2	1.2	1.7	2.1	2.5
3	1.4	2.1	2.6	3.0
4	1.5	2.4	3.0	3.6

(correct digits measured using (4.3))

TABLE 5
The 3-level symmetric scheme for (4.1) *with* $\lambda = -36$

r/m	1	2	3	4
1	.9	1.2	1.5	1.7
2	1.2	1.7	2.1	2.5
3	1.4	2.1	2.6	3.0
4	1.5	2.4	3.0	3.6

(correct digits measured using (4.3)).

symmetric and nonsymmetric multilevel schemes, as a function of $1 \leqq m \leqq 4$ and $1 \leqq r \leqq 4$. Several points are worth noting.

First, in the case of the 4-level scheme ($\mathcal{T}_1$ is 3×3), divergence of the nonsymmetric scheme is observed for $m \geqq 4$, illustrating the phenomenon predicted by (2.22). The problem can be corrected by making h_1 smaller, i.e., using a 3-level scheme with $\mathcal{T}_1$ the 5×5 mesh. In contrast to the nonsymmetric scheme, the symmetric 4-level scheme converges for all values of m.

Second, the symmetric 3-level scheme converges more slowly than the nonsymmetric scheme, illustrating the effect of the squared condition number in the symmetric scheme. Although convergence to u_h is slower, note that u_h has only 1.08 digits of accuracy relative to u, so that taking m and r to be 1-2 is adequate for both methods in terms of computing an approximation of u.

A timing study of the 3-level symmetric scheme as a function of m and r is given in Table 6. The time appears to be a bilinear function of m and r, with the cost of increasing

TABLE 6
Timing of 3-level symmetric scheme for (4.1)

r/m	1	2	3	4
1	.2	.27	.33	.39
2	.39	.52	.64	.76
3	.57	.77	.95	1.13
4	.75	1.00	1.25	1.50

(timings measured in seconds on a CDC 6600 (MNF compiler))

r somewhat larger than the cost of increasing m, as suggested by (2.17). The timing differences between symmetric and nonsymmetric schemes for this problem (289 unknowns) were insignificant.

When we tried both schemes for various values of λ the following behavior was observed: on $(0, -2\pi^2)$, both schemes work well since the operator and all matrices are symmetric and positive definite. For $-26 \lessapprox \lambda < -2\pi^2$ the 4-level scheme did not converge in either its symmetric or nonsymmetric variants. This is because the 9×9 matrix corresponding to the 3×3 grid $\mathcal{T}_1$ remains positive definite until $\lambda \approx -24$. For this range of λ, the 3×3 grid is not sufficiently fine to approximate even the smoothest components of the solution (on $(-2\pi^2, -24)$, u_1 of (1.9) does not even agree with u in sign).

The problem can be corrected by making h_1 smaller. The matrices corresponding to the 5×5, 9×9, and 17×17 meshes are singular at $\lambda \approx -21.66$, -20.27, and -19.88 respectively. Thus the 5×5 mesh failed to adequately approximate the smooth components of the solution on a shorter interval in λ, approximately $-23 \lessapprox \lambda < -2\pi^2$. The 2-level scheme ($\mathcal{T}_1$ the 9×9 mesh) converged for $\lambda = -21$. Similar sorts of behavior can be expected in the vicinity of other eigenvalues.

We next consider the problem

$$\begin{aligned} -\Delta u + \beta u_x &= f \quad \text{in } \Omega = (0, 1) \times (0, 1), \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

with true solution $u = \sin(\pi x) \sin(\pi y)$. We gave PLTMG the 3×3 mesh of Fig. 1, and generated three additional meshes as described in the first example. We solved this

problem using both symmetric and nonsymmetric schemes, 2, 3, and 4 levels, for $\beta = 10^j$, $-2 \leqq j \leqq 2$.

In Tables 7 and 8 we present results, for $\beta = 100$, of the 3-level symmetric scheme and the 2-level nonsymmetric scheme. The nonsymmetric scheme has behavior qualitatively similar to that which is displayed in the first example (Table 1). To solve

TABLE 7
The 2-level nonsymmetric scheme for (4.4) *with* $\beta = 100$

r/m	1	2	3	4
1	1.1	.9	.6	.4
2	1.2	.8	.3	−.2
3	1.3	.5	−.2	−.9
4	1.2	.2	−.7	−1.5

(correct digits measured using (4.3))

TABLE 8
The 3-level symmetric scheme for (4.4) *with* $\beta = 100$

r/m	1	2	3	4
1	.9	1.2	1.4	1.5
2	1.2	1.7	2.0	2.5
3	1.3	2.1	2.8	3.3
4	1.5	2.5	3.3	4.0

(correct digits measured using (4.3))

(4.4) with $\beta = 100$ using this nonsymmetric scheme appears to require $\mathcal{T}_1$ to be finer than 9×9 to get convergence for the range of m and r studied. In contrast, the 3-level symmetric scheme does quite nicely. The 4-level symmetric scheme did not converge for $\beta = 100$ for all values of m and r tested.

Generally speaking, the effect of the squared condition number for the symmetric scheme is not as severe in this problem. On the range of β tested, the finite element solution on the 17×17 grid had 1.07–1.09 digits of accuracy measured using (4.2). With the exception of $\beta = 100$ for 4-levels, convergence of the symmetric scheme was similar to that of Table 8 irrespective of the size of β or the number of levels.

In contrast, using the nonsymmetric scheme, convergence similar to that of Table 2 was recorded for small values of β. As β increased, one had to increase the fineness of $\mathcal{T}_1$ (decrease the number of levels) in order to obtain convergence, and even then it was comparable to or worse than the symmetric scheme. The nonsymmetric scheme overflowed for $\beta = 100$ using 4 levels.

Appendix. Proof of Lemma 1.

The case $\nu = \psi$. To prove (3.7) we must show (3.7) for the cases $s = 0$ and $s = 1$. The remaining cases then follow from the interpolation of norms argument given in the appendix of [3]. The case $s = 0$ is trivial, and clearly

$$C^{-1}(a)\|\chi\|_1 \leqq |||\chi|||_{1,\psi} \leqq C(a)\|\chi\|_1,$$

and so (3.7) follows.

To prove (3.8), we use an argument due to Thomeé [13]. Let $\eta_h \in \mathcal{M}_j$ and $\eta \in \mathcal{H}^1(\Omega)$ be solutions of

$$\text{(A1)} \qquad \begin{aligned} a_1(\eta_h, v) &= (\chi, v), \quad \text{for all } v \in \mathcal{M}_j, \\ a_1(\eta, v) &= (\chi, v), \quad \text{for all } v \in \mathcal{H}^1(\Omega). \end{aligned}$$

Note

$$\text{(A2)} \qquad a_1(\eta - \eta_h, v) = 0, \quad \text{for all } v \in \mathcal{M}_j.$$

If $\chi = \sum_{k=1}^{N_j} c_k \psi_k$, then $\eta_h = \sum_{k=1}^{N_j} c_k \lambda_k^{-1} \psi_k$ and

$$\text{(A3)} \qquad |||\chi|||_{-(1+\alpha),\psi} = |||\eta_h|||_{1-\alpha,\psi} \leqq C \|\eta_h\|_{1-\alpha},$$

by virtue of (3.7). Now

$$\text{(A4)} \qquad \|\eta_h\|_{1-\alpha} \leqq \|\eta\|_{1-\alpha} + \|\eta_h - \eta\|_{1-\alpha}.$$

By elliptic regularity (1.5),

$$\text{(A5)} \qquad \|\eta\|_{1-\alpha} \leqq C \|\chi\|_{-(1+\alpha)}.$$

To estimate $\|\eta_h - \eta\|_{1-\alpha}$ use a standard duality argument; let ζ, ξ satisfy

$$a_1(\zeta, v) = (\xi, v), \quad \text{for all } v \in \mathcal{H}^1(\Omega).$$

Take $v = \eta_h - \eta$ and use (A2), (1.9) and then (1.5) to see

$$\text{(A6)} \qquad \begin{aligned} (\xi, \eta - \eta_h) &= a_1(\zeta - v, \eta - \eta_h) \\ &\leqq C h_j^{\alpha} \|\zeta\|_{1+\alpha} \|\eta - \eta_h\|_1 \\ &\leqq C h_j^{\alpha} \|\xi\|_{\alpha-1} \|\eta - \eta_h\|_1, \qquad v \in \mathcal{M}_j. \end{aligned}$$

Thus, by virtue of (1.9), (A6) and (1.5) once more,

$$\text{(A7)} \qquad \begin{aligned} \|\eta - \eta_h\|_{1-\alpha} &\leqq C h_j^{\alpha} \|\eta - \eta_h\|_1 \\ &\leqq C h_j^{2\alpha} \|\eta\|_{1+\alpha} \\ &\leqq C h_j^{2\alpha} \|\chi\|_{\alpha-1}. \end{aligned}$$

Equation (3.8) then follows from the estimates (A3), (A4), (A5) and (A7).

The case $v = \varphi$. To prove Lemma 1 for $\nu = \varphi$, we prove there exists a constant $C = C(a, b, c, \Omega, \delta_0, \delta_1, h_1)$ such that, for all $\chi \in \mathcal{M}_j$,

$$\text{(A8)} \qquad C^{-1} |||\chi|||_{s,\psi} \leqq |||\chi|||_{s,\varphi} \leqq C |||\chi|||_{s,\psi}, \qquad -2 \leqq s \leqq 2.$$

Inequalities (3.7)–(3.8) for $\nu = \varphi$ follow immediately from the corresponding equalities for $\nu = \psi$ and (A8).

First note (A8) holds trivially for the case $s = 0$. We will prove (A8) for the cases $s = \pm 2$; the interpolation of norms argument given in the appendix of [3] then shows (A8) holds for $-2 \leqq s \leqq 2$.

Let A, A_1 and B be defined as in § 3 and let $X \in R^{N_j}$ correspond to $\chi \in \mathcal{M}_j$. Showing the right-hand side of (A8) for $s = 2$ is equivalent to showing

$$\text{(A9)} \qquad \sup_{X \in R^{N_j}} \frac{X^T A^T A X}{X^T A_1^2 X} \leqq C.$$

Letting $A = A_1 + B$, we have

$$\frac{X^T A^T A X}{X^T A_1^2 X} = \frac{X^T (A_1^2 + A_1 B + B^T A_1 + B^T B) X}{X^T A_1^2 X} \leqq 2 + 2\frac{\|BX\|_{l_2}^2}{\|A_1 X\|_{l_2}^2}.$$

Let $\{\rho_k\}$ be any basis for $\mathcal{M}_j$ which is orthonormal with respect to the $\mathcal{L}_2(\Omega)$ inner product. Then $(b\nabla\chi + (c-1)\chi, \rho_k)$ is just the kth coefficient of the $\mathcal{L}_2(\Omega)$ projection of $b\nabla\chi + (c-1)\chi$ into $\mathcal{M}_j$. Thus, for the basis $\{\rho_k\}$,

$$\text{(A10)} \quad \frac{\|BX\|_{l_2}^2}{\|A_1 X\|_{l_2}^2} = \sum_{k=1}^{N_j} \frac{(b\nabla\chi + (c-1)\chi, \rho_k)^2}{|||\chi|||_{2,\psi}^2} \leqq \frac{\|b\nabla\chi + (c-1)\chi\|_0^2}{|||\chi|||_{2,\psi}^2} \leqq C\frac{|||\chi|||_{1,\psi}^2}{|||\chi|||_{2,\psi}^2} \leqq C,$$

verifying (A9).

To show the left side of (A8) for $s = 2$, we must show

$$\text{(A11)} \quad \sup_{X \in R^{N_j}} \frac{X^T A_1^2 X}{X^T A^T A X} \leqq C.$$

Letting $A_1 = A - B = A^T - B^T$, we see that (A11) can be reduced to bounding $\|BX\|_{l_2}^2 / \|AX\|_{l_2}^2$. To do this we first prove that for any $\chi \in \mathcal{M}_j$ there exists $\rho \in \mathcal{M}_j$ such that

$$\text{(A12)} \quad |a(\chi, \rho)| \geqq C\|\chi\|_1, \qquad \|\rho\|_0 = 1.$$

To do this, we must essentially verify a discrete inf-sup condition on the bilinear form $a(\cdot, \cdot)$ [1]. Note that the Garding inequality [11]

$$a(\chi, \chi) = a_1(\chi, \chi) + b(\chi, \chi) \geqq C\|\chi\|_1^2 - C'\|\chi\|_0^2$$

implies there exists a $\lambda \geqq 0$, $\lambda = \lambda(a, b, c, \delta_0, \delta_1, \Omega, h_1)$ such that

$$a(\chi, \chi) + \lambda(\chi, \chi) \geqq C\|\chi\|_1^2.$$

Let ξ and ξ_h satisfy

$$a^*(\xi, v) = \lambda(\chi, v), \qquad v \in \mathcal{H}^1(\Omega),$$
$$a^*(\xi_h, v) = \lambda(\chi, v), \qquad v \in \mathcal{M}_j.$$

By elliptic regularity (1.5) and the approximation property (1.8), we have

$$\|\xi - \xi_h\|_1 \leqq Ch_j^\alpha \|\xi\|_{1+\alpha} \leqq Ch_j^\alpha \|\chi\|_{\alpha-1} \leqq Ch_j^\alpha \|\chi\|_1,$$
$$\|\xi\|_1 \leqq C\|\chi\|_{-1} \leqq C\|\chi\|_1.$$

Thus

$$\|\chi + \xi_h\|_1 \leqq \|\chi\|_1 + \|\xi_h - \xi\|_1 + \|\xi\|_1 \leqq C\|\chi\|_1$$

and

$$a(\chi, \chi + \xi_h) = a(\chi, \chi) + a(\chi, \xi_h) = a(\chi, \chi) + \lambda(\chi, \chi) \geqq C\|\chi\|_1^2 \geqq C\|\chi\|_1 \|\chi + \xi_h\|_1.$$

If $\rho = (\chi + \xi_h)/\|\chi + \xi_h\|_0$, then

$$a(\chi, \rho) \geqq C\|\chi\|_1 \frac{\|\chi + \xi_h\|_1}{\|\chi + \xi_h\|_0} \geqq C\|\chi\|_1,$$

verifying (A12).

Starting from $\rho_1 = \rho$, complete an orthonormal basis $\{\rho_k\}$ for $\mathcal{M}_j$ with respect to the $\mathcal{L}_2(\Omega)$ inner product.

Use (A12) and the basis $\{\rho_k\}$,

$$\text{(A13)}\qquad \frac{\|BX\|_{l_2}^2}{\|AX\|_{l_2}^2}=\frac{\sum_{k=1}^{N_j}(b\cdot\nabla\chi+(c-1)\chi,\rho_k)^2}{\sum_{k=1}^{N_j}a(\chi,\rho_k)^2}\leqq C\frac{\|b\cdot\nabla\chi+(c-1)\chi\|_0^2}{\|\chi\|_1^2}\leqq C,$$

verifying (A11).

Next, to show the right-hand side of (A8) for $s=-2$ we will show

$$\text{(A14)}\qquad \sup_{X\in R^{N_j}}\frac{X^T(A^TA)^{-1}X}{X^TA_1^{-2}X}\leqq C.$$

Letting $Y=A_1^{-1}X$ and using $A_1=A-B$, we have

$$\frac{X^T(A^TA)^{-1}X}{X^TA_1^{-2}X}=\frac{Y^T(A_1A^{-1})(A_1A^{-1})^TY}{Y^TY}\leqq 2+2\|BA^{-1}\|_{l_2}^2.$$

Now, letting $X=AY$, we have

$$\|BA^{-1}\|_{l_2}^2=\sup_{Y\in R^{N_j}}\frac{Y^TA^{-T}B^TBA^{-1}Y}{Y^TY}=\sup_{X\in R^{N_j}}\frac{\|BX\|_{l_2}^2}{\|AX\|_{l_2}^2}.$$

The estimate (A13) now establishes (A14).

Finally, to verify the right-hand side of (A8) for $s=-2$, we must prove

$$\text{(A15)}\qquad \sup_{X\in R^{N_j}}\frac{X^TA_1^{-2}X}{X^T(A^TA)^{-1}X}\leqq C.$$

But (A15) can be reduced to bounding $\|BX\|_{l_2}^2/\|A_1X\|_{l_2}^2$, which was done in (A10).

The case $\nu=\theta$. To prove Lemma 1 for $\nu=\theta$, we will show that for $\chi\in\mathcal{M}_j$

$$\text{(A16)}\qquad C^{-1}|||\chi|||_{s,\psi}\leqq|||\chi|||_{s,\theta}\leqq C|||\chi|||_{s,\psi},\qquad -2\leqq s\leqq 2.$$

The same series of arguments used for the case $\nu=\varphi$ reduces the problem to bounding

$$\|B^TX\|_{l_2}^2/\|A_1X\|_{l_2}^2\quad\text{and}\quad\|B^TX\|_{l_2}^2/\|A^TX\|_{l_2}^2.$$

Let $X,\ Y\in R^{N_j}$ correspond respectively to $\chi,\ \rho\in\mathcal{M}_j$. Then

$$\begin{aligned}Y^TB^TX=(b\cdot\nabla\rho+(c-1)\rho,\chi)&=(-\nabla\cdot(b\chi)+(c-1)\chi,\rho)+\langle b\cdot n\chi,\rho\rangle\\&=b_1(\chi,\rho)+b_0(\chi,\rho),\end{aligned}$$

where

$$\langle u,v\rangle=\int_{\partial\Omega}uv\,dx,$$

$$|u|_0^2=\langle u,u\rangle.$$

The $b_1(\cdot,\cdot)$ inner product is of the type we have analyzed in the case $\nu=\varphi$. Let $B^T=B_1+B_0$ where B_i corresponds to $b_i(\cdot,\cdot)$, $i=0,1$. We seek a bound for $\|B_0X\|_{l_2}^2$. To do this let $\{\xi_k\}$ be the nodal basis for $\mathcal{M}_j$. Take those basis functions with nontrivial traces on $\partial\Omega$ and orthonormalize them with respect to the $\langle\cdot,\cdot\rangle$ inner product, obtaining a basis $\{\rho_k\}$. Then the nonzero coefficients among $\langle b\cdot\eta\chi,\rho_k\rangle$ are the coefficients of the $\mathcal{L}_2(\partial\Omega)$ projection of $b\cdot n\chi$ into the subspace of $\mathcal{M}_j$ spanned by the nodal basis functions with nontrivial traces on $\partial\Omega$. Thus

$$\text{(A17)}\qquad \|B_0X\|_{l_2}^2=\sum_k^{N_j}\langle b\cdot n,\rho_k\rangle^2\leqq|b\cdot n\chi|_0^2\leqq C\|\chi\|_1^2.$$

To bound $\|B^T X\|_{l_2}^2 / \|A_1 X\|_{l_2}^2$, we have

$$\frac{\|B^T X\|_{l_2}^2}{\|A_1 X\|_{l_2}^2} \leqq C\left\{\frac{\|B_1 X\|_{l_2}^2}{\|A_1 X\|_{l_2}^2} + \frac{\|B_0 X\|_{l_2}^2}{\|A_1 X\|_{l_2}^2}\right\} \leqq C \frac{|||\chi|||_{1,\psi}^2}{|||\chi|||_{2,\psi}^2} \leqq C,$$

where we have used an argument similar to (A10) on the first term, and (A17) and (3.7) for $\nu = \psi$, $s = 1$ for the second.

To bound $\|B^T X\|_{l_2}^2 / \|A^T X\|_{l_2}^2$, we use an argument similar to that used in proving (A12) to show that for any $\chi \in \mathcal{M}_j$ there exists $\rho \in \mathcal{M}_j$ such that

$$|a^*(\chi, \rho)| = |a(\rho, \chi)| \geqq C\|\chi\|_1, \qquad \|\rho\|_0 = 1. \tag{A18}$$

Then

$$\frac{\|B^T X\|_{l_2}^2}{\|A^T X\|_{l_2}^2} \leqq C\left\{\frac{\|B_1 X\|_{l_2}^2}{\|A^T X\|_{l_2}^2} + \frac{\|B_0 X\|_{l_2}^2}{\|A^T X\|_{l_2}^2}\right\} \leqq C \frac{|||\chi|||_{1,\psi}^2}{|||\chi|||_{1,\psi}^2} \leqq C,$$

where we have used an argument similar to (A13) on the first term and (A17)–(A18) on the second.

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REFERENCES

[1] I. BABUSKA AND A. K. AZIZ, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Academic Press, New York, 1972.

[2] N. S. BAKHVALOV, *On the convergence of a relaxation method with natural constraints on the elliptic operator*, Zh. Vychisl. Mat. mat. Fiz., 6(1966), pp. 861–885.

[3] R. BANK AND T. DUPONT, *An optimal order process for solving finite element equations*, Math. Comp., 36 (1981).

[4] R. BANK AND A. SHERMAN, *Algorithmic aspects of the multi-level solution of finite element equations*, in Sparse Matrix Proceedings, 1978, I.S. Duff and G. W. Stewart, eds., Society for Industrial and Applied Mathematics, Philadelphia, 1979.

[5] A. BRANDT, *Multi-level adaptive solutions to boundary value problems*, Math. Comp., 31(1977), pp. 333–390.

[6] ———, *Multi-level adaptive techniques for partial differential equations*, in Ideas and Software Mathematical Software III, Academic Press, New York, 1977.

[7] W. HACKBUSCH, *On the convergence of a multi-grid iteration applied to finite element equations*, Report 77-8, Universitat zu Koln, July, 1977.

[8] R. A. NICOLAIDES, *On the l^2 convergence of an algorithm for solving finite element equations*, Math. Comp., 140(1977), pp. 892–906.

[9] ———, *On multi-grid convergence in the indefinite case*, Math. Comp., 32(1978), pp. 1082–1086.

[10] D. J. ROSE AND G. F. WHITTEN, *An analysis of dissection strategies*, in Sparse Matrix Computations, Academic Press, New York, pp. 59–83.

[11] A. H. SCHATZ, *An observation concerning Ritz-Galerkin methods with indefinite bilinear forms*, Math. Comp., 28(1974), pp. 959–962.

[12] G. STRANG AND G. FIX, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[13] V. THOMEÉ, *Negative norm estimates and superconvergence in Galerkin methods for parabolic problems*, Math. Comp., 34(1980), pp. 93–114.