

# A COMPARISON BETWEEN THE MINI-ELEMENT AND THE PETROV-GALERKIN FORMULATIONS FOR THE GENERALIZED STOKES PROBLEM

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**Abstract.** We first derive the statically condensed form of the mini-element formulation for the generalized Stokes equations, then present a Petrov-Galerkin-like formulation for the same problem. We show that the two are closely related.

**Key words.** Mixed finite element methods, generalized Stokes equations, mini-element, Petrov-Galerkin formulation.

**AMS subject classifications.** 65F10, 65N20, 65N30.

**1. Introduction.** We consider the *generalized Stokes problem*

$$(1) \quad \begin{cases} \alpha \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} & \text{on } \partial\Omega \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathcal{R}^2$ . The function  $\mathbf{f} = (f_1, f_2)$  is a smooth function,  $\mathbf{g}$  is piecewise linear and satisfies the compatibility condition  $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} ds = 0$ . Furthermore  $\int_{\Omega} p \, d\Omega$  is assumed to be 0. The constant  $\nu$  is a viscosity parameter. The term  $\alpha \mathbf{u}$  typically comes from a classical Euler time discretization of the full Navier-Stokes equations ( $\alpha$  being proportional to the inverse of the time step  $\Delta t$ ) in conjunction with the use of a  $\theta$ -scheme. Both  $\nu$  and  $\alpha$  are positive.

A mixed formulation of (1) is given by the finding of  $(\mathbf{u}, p) \in \mathcal{H}_g^1(\Omega) \times \mathcal{L}_0^2(\Omega)$  such that

$$(2) \quad \begin{cases} \alpha(\mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \mathbf{v}) &= (f, \mathbf{v}) & \mathbf{v} \in \mathcal{H}_0^1(\Omega) \\ -(q, \nabla \cdot \mathbf{u}) &= 0 & p \in \mathcal{L}_0^2(\Omega) \end{cases}$$

where  $\mathcal{H}_g^1(\Omega) \equiv \{\mathbf{u} \in (\mathcal{H}^1)^2, \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\}$ .  $(\cdot, \cdot)$  represents the usual  $\mathcal{L}^2$  inner product.

In this work we will compare two popular discretizations for (1), the mini-element discretization of Arnold, Brezzi and Fortin [1] and the Petrov-Galerkin scheme of Hughes, Franca and Balestra [4] [3]. Although initially dissimilar in approach, we show that the two discretizations lead to strikingly similar sets of linear equations when the bubble unknowns of the mini-element formulation are statically condensed, and when piecewise linear velocities and pressures are chosen for the Petrov-Galerkin scheme.

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In particular, when  $\alpha = 0$  (Stokes equations) the matrices are identical, for a proper choice of the elementwise stability constants in the Petrov-Galerkin scheme. The right-hand sides differ only in terms formally of order  $\mathcal{O}(h_\tau^2)$ . When  $\alpha > 0$ , the matrices are no longer identical, but differ by only a small amount.

In sections 2 and 3 we will discuss the mini-element and the Petrov-Galerkin discretizations respectively. In section 4 we will analyze the similarities and differences between the two formulations.

Let  $\mathcal{T}$  be a triangulation of  $\Omega$  such that any two triangles in  $\mathcal{T}$  share at most a vertex or an edge. For  $\tau \in \mathcal{T}$  let  $h_\tau$  be the diameter of  $\tau$ . Let  $h = \max_{\tau \in \mathcal{T}} h_\tau$ .

For  $\tau \in \mathcal{T}$  let also  $\psi_i(\tau)$ ,  $i = 1, 3$  be the barycentric coordinates (nodal basis functions). In the following  $\psi_i(\tau)$  will be replaced by  $\psi_i$  for simplicity and when no confusion is possible.

In the following we suppose without loss of generality that  $\mathbf{g} = \mathbf{0}$ . Then we define the spaces

$$\mathcal{L}_\tau \equiv \text{span}\{\psi_i, 1 \leq i \leq 3\}$$

$$\mathcal{K}_\tau \equiv \text{span}\{\psi_b = \psi_1\psi_2\psi_3\}$$

and the corresponding global spaces

$$\mathcal{L} = \prod_{\tau \in \mathcal{T}} \mathcal{L}_\tau, \quad \mathcal{K} = \prod_{\tau \in \mathcal{T}} \mathcal{K}_\tau \quad \text{and} \quad \mathcal{L}_0 = \mathcal{L} \cap \mathcal{H}_0^1$$

$\mathcal{L}_0$  is the space of continuous piecewise linear functions on  $\mathcal{T}$  and  $\mathcal{K}$  is the space of cubic "bubble" functions on  $\mathcal{T}$ .

**2. The mini-element formulation.** The solution  $(\mathbf{u}, p)$  of (2) is approximated by the solution  $(\mathbf{u}_h, p_h) \in (\mathcal{L}_0 \oplus \mathcal{K})^2 \times \mathcal{L}$  of the discrete equivalent of (2):

$$(3) \quad \begin{cases} \alpha(\mathbf{u}_h, \mathbf{v}) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) &= (f, \mathbf{v}) & \mathbf{v} \in (\mathcal{L}_0 \oplus \mathcal{K})^2 \\ -(q, \nabla \cdot \mathbf{u}_h) &= 0 & q \in \mathcal{L} \end{cases}$$

This formulation is known to satisfy a Babuška-Brezzi condition and thus to produce a unique and stable solution  $(\mathbf{u}_h, p_h)$ . This solution can be uniquely decomposed into its linear part  $(\mathbf{u}_{h,l}, p_h)$  and its bubble part  $(\mathbf{u}_{h,b}, 0)$ . In practice  $(\mathbf{u}_{h,l}, p_h)$  is considered a better solution than  $(\mathbf{u}_h, p_h)$  itself [2] [5].

Using elementwise integration, the contribution of a triangle  $\tau \in \mathcal{T}$  is then defined by the following  $11 \times 11$  matrix  $M_\tau$  (4 unknowns for each component of the velocity and 3 for the pressure):

$$(4) \quad M_\tau = |\tau| \begin{pmatrix} A_\tau & \mathbf{e}_\tau & 0 & 0 & B_{\tau,x}^t \\ \mathbf{e}_\tau^t & \sigma'_\tau & 0 & 0 & \mathbf{w}_{\tau,x}^t \\ 0 & 0 & A_\tau & \mathbf{e}_\tau & B_{\tau,y}^t \\ 0 & 0 & \mathbf{e}_\tau^t & \sigma'_\tau & \mathbf{w}_{\tau,y}^t \\ B_{\tau,x} & \mathbf{w}_{\tau,x} & B_{\tau,y} & \mathbf{w}_{\tau,y} & 0 \end{pmatrix}$$

Here  $|\tau|$  is the area of the triangle  $\tau$ , the  $3 \times 3$  matrices  $A_\tau$ ,  $B_{\tau,x}$ , and  $B_{\tau,y}$  correspond to inner products involving linear basis function for the velocity and the pressure, the scalar  $\sigma'_\tau$  is the contribution to the  $\mathcal{H}^1$  inner product from the cubic bubble functions

for the velocity, and the 3-vectors  $\mathbf{w}_{\tau,x}$  and  $\mathbf{w}_{\tau,y}$  correspond to contributions to the divergence term for bump velocity basis functions and linear pressure basis functions.

Direct calculation of the relevant integrals, along with some algebraic manipulation, yields

$$(5) \quad (A_\tau)_{i,j} = \nu \nabla \psi_i \cdot \nabla \psi_j + \frac{\alpha}{12} (1 + \delta_{ij}) \quad 1 \leq i, j \leq 3$$

$$(6) \quad (B_{\tau,x})_{i,j} = -\frac{1}{3} \psi_{j,x} \quad \text{and} \quad (B_{\tau,y})_{i,j} = -\frac{1}{3} \psi_{j,y} \quad 1 \leq i, j \leq 3$$

$$(7) \quad \mathbf{w}_{\tau,x} = \frac{1}{60} \begin{pmatrix} \psi_{1,x} \\ \psi_{2,x} \\ \psi_{3,x} \end{pmatrix} \quad \mathbf{w}_{\tau,y} = \frac{1}{60} \begin{pmatrix} \psi_{1,y} \\ \psi_{2,y} \\ \psi_{3,y} \end{pmatrix} \quad \mathbf{e}_\tau = \frac{\alpha}{180} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$(8) \quad \sigma'_\tau = \sigma_\tau \nu + \frac{\alpha}{2520}$$

Here  $\delta_{ij}$  is the usual Kronecker symbol,  $\psi_{i,x}$  or  $\psi_{i,y}$  denotes the partial derivative of  $\psi_i$  with respect to  $x$  or  $y$ , and the quantity  $\sigma_\tau$  is defined by  $\sigma_\tau = \frac{1}{|\tau|} (\nabla \psi_b(\tau), \nabla \psi_b(\tau))_\tau$ .

The contribution of the triangle  $\tau$  to the right-hand side of (3) is an 11-vector  $\mathbf{r}$  such that

$$(\mathbf{r}_\tau)_i = \begin{cases} (f_1, \psi_i)_\tau & 1 \leq i \leq 3 \\ (f_1, \psi_b)_\tau & i = 4 \\ (f_2, \psi_{i-4})_\tau & 5 \leq i \leq 7 \\ (f_2, \psi_b)_\tau & i = 8 \\ 0 & 9 \leq i \leq 11 \end{cases}$$

A simple calculation shows that after elimination of the bubble unknowns one gets the following  $9 \times 9$  elementary contribution for the triangle  $\tau$ :

$$(9) \quad M'_\tau = |\tau| \begin{bmatrix} A'_\tau & 0 & B'^t_{\tau,x} \\ 0 & A'_\tau & B'^t_{\tau,y} \\ B'_{\tau,x} & B'_{\tau,y} & -S'_\tau \end{bmatrix}$$

where

$$(10) \quad A'_\tau = A_\tau - \frac{1}{\sigma'_\tau} \mathbf{e}_\tau \mathbf{e}_\tau^t$$

$$(11) \quad B'_{\tau,x} = B_{\tau,x} - \frac{1}{\sigma'_\tau} \mathbf{w}_{\tau,x} \mathbf{e}_\tau^t$$

$$(12) \quad B'_{\tau,y} = B_{\tau,y} - \frac{1}{\sigma'_\tau} \mathbf{w}_{\tau,y} \mathbf{e}_\tau^t$$

$$(13)$$

and the  $3 \times 3$  "stability" matrix  $S_\tau$  is defined by

$$(14) \quad S_\tau = \frac{1}{\sigma'_\tau} (\mathbf{w}_{\tau,x} \mathbf{w}_{\tau,x}^t + \mathbf{w}_{\tau,y} \mathbf{w}_{\tau,y}^t) = \frac{1}{3600 \sigma'_\tau} (\nabla \psi_i, \nabla \psi_j)_{1 \leq i, j \leq 3}$$

In the important case of the regular Stokes equations ( $\alpha = 0$ ) the bubble functions contribute only two diagonal terms  $\sigma_\tau$  in  $A_\tau$  (one for each component of the velocity), and the matrix  $S_\tau$  is the only difference between the mini-element formulation and a formulation using continuous piecewise linear functions for both velocity and pressure, whence the name of "stability" matrix, since the latter formulation is not admissible and does not satisfy an inf-sup condition.

The condensed right-hand side now reads

$$(\mathbf{r}'_\tau)_i = \begin{cases} (f_1, \psi_i)_\tau - \frac{\alpha}{180\sigma'_\tau}(f_1, \psi_b)_\tau & 1 \leq i \leq 3 \\ (f_2, \psi_{i-3})_\tau - \frac{\alpha}{180\sigma'_\tau}(f_2, \psi_b)_\tau & 4 \leq i \leq 6 \\ -\frac{1}{60\sigma'_\tau}(\mathbf{f}, \psi_b \nabla \psi_{i-6})_\tau & 7 \leq i \leq 9 \end{cases}$$

All elemental condensed matrices and right-hand sides are finally assembled in the global stiffness matrix and global right-hand side. The resulting system (modulo modifications due to the boundary conditions) has then a unique solution, which is the nodal values of  $(\mathbf{u}_{h,l}, p_h)$ .

**3. The Petrov-Galerkin formulation.** T.J.Hughes *et al* [4] first proposed to modify the saddlepoint problem (2) in order to forego the imposition of an inf-sup or related condition. The idea behind the resulting *Petrov-Galerkin* formulation is the introduction of some kind of "stability" matrix to the divergence equation in (1), by adding to it a multiple of order  $\mathcal{O}(h_\tau^2)$  of the first equation integrated against a special test function elementwise. In this section we extend this technique to the generalized Stokes equations. In particular, we want to find the solution  $(\mathbf{u}_L, p_L)$  of the following system, for  $(\mathbf{v}, q) \in \mathcal{L}_0 \times \mathcal{L}_0 \times \mathcal{L}$ :

$$(15) \quad \left\{ \begin{array}{l} \alpha(\mathbf{u}_L, \mathbf{v}) + \nu(\nabla \mathbf{u}_L, \nabla \mathbf{v}) - (p_L, \nabla \cdot \mathbf{v}) \\ - \sum_{\tau \in \mathcal{T}} \alpha \lambda_\tau (\alpha \mathbf{u}_L - \nu \Delta \mathbf{u}_L + \nabla p_L, \mathbf{v})_\tau = (f, \mathbf{v}) - \sum_{\tau \in \mathcal{T}} \alpha \lambda_\tau (\mathbf{f}, \mathbf{v})_\tau \\ \hspace{15em} \mathbf{v} \in \mathcal{L}_0 \times \mathcal{L}_0 \\ - (q, \nabla \cdot \mathbf{u}_L) - \sum_{\tau \in \mathcal{T}} \lambda_\tau (\alpha \mathbf{u}_L - \nu \Delta \mathbf{u}_L + \nabla p_L, \nabla q)_\tau \\ \hspace{15em} = - \sum_{\tau \in \mathcal{T}} \lambda_\tau (\mathbf{f}, \nabla q)_\tau \hspace{2em} q \in \mathcal{L} \end{array} \right.$$

The first equation results from adding to the first equation of (3) a sum over triangles  $\tau$  of a positive multiple  $\lambda_\tau$  (typically of order  $\mathcal{O}(h_\tau^2)$ ) of the same equation integrated over  $\tau$ . A similar manipulation is done on the second equation, with the difference that the added part is now integrated against  $\nabla q$ . Note also that  $\Delta \mathbf{u}_L = 0$ . The resulting stiffness matrix is then symmetric. In the more general case of higher degree interpolation polynomials, this term would result in a non-symmetric contribution to the stiffness matrix.

Moreover, the bilinear form

$$B((\mathbf{u}, p), (\mathbf{v}, q)) \equiv \alpha(\mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) + \sum_{\tau \in \mathcal{T}} \lambda_\tau (\nabla p + \alpha \mathbf{u}, \nabla q - \alpha \mathbf{v})_\tau$$

is coercive over  $\mathcal{L}_0 \times \mathcal{L}_0 \times \mathcal{L}$  for the norm

$$\|(\mathbf{u}, p)\|_*^2 = \alpha \|\bar{\mathbf{u}}\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \sum_{\tau \in \mathcal{T}} \lambda_\tau \|\nabla p\|_\tau^2$$

provided that  $\alpha \lambda_\tau < 1$  for all  $\tau \in \mathcal{T}$ .

Because of (8) the particular case  $\lambda_\tau = \frac{1}{3600\sigma'_\tau}$  satisfies this condition. With this choice, the coercivity of  $B(\cdot, \cdot)$  immediately implies the unique solvability of (15).

The corresponding element stiffness matrix results then in the following  $9 \times 9$  matrix:

$$(16) \quad M'_\tau = |\tau| \begin{bmatrix} A''_\tau & 0 & B'^t_{\tau,x} \\ 0 & A''_\tau & B'^t_{\tau,y} \\ B'_{\tau,x} & B'_{\tau,y} & -S_\tau \end{bmatrix}$$

where the matrices  $B'_{\tau,x}, B'_{\tau,y}$  and  $S_\tau$  were defined in section 2 and

$$(17) \quad A''_\tau = A_\tau - \frac{1}{\sigma'_\tau} D_\tau$$

$$(18) \quad D_\tau = \frac{\alpha^2}{3600} \frac{1}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

The corresponding elemental right-hand side reads:

$$(\mathbf{r}''_\tau)_i = \begin{cases} \left(1 - \frac{\alpha}{3600\sigma'_\tau}\right) (f_1, \psi_i)_\tau & 1 \leq i \leq 3 \\ \left(1 - \frac{\alpha}{3600\sigma'_\tau}\right) (f_2, \psi_{i-3})_\tau & 4 \leq i \leq 6 \\ -\frac{1}{3600\sigma'_\tau} (\mathbf{f}, \nabla \psi_{i-6})_\tau & 7 \leq i \leq 9 \end{cases}$$

These results are very close from the quantities we obtained in section 2 for the mini-element formulation. In the next section we compare both formulations and derive an estimate for  $\|(\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L)\|_*$ .

**4. Comparison of the formulations.** The element stiffness matrices  $M'_\tau$  and  $M''_\tau$  are very similar for the choice of  $\lambda_\tau$  made in section 3. Indeed they differ only by the quantity  $\frac{1}{\sigma'_\tau} \mathbf{e}_\tau \mathbf{e}_\tau^t$  (resp.  $\frac{1}{\sigma'_\tau} D_\tau$ ). Furthermore, we have

$$\mathbf{e}_\tau \mathbf{e}_\tau^t = \frac{\alpha^2}{3600} \frac{1}{9} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

which corresponds to the application of the one-point (triangle's barycenter) rule integration, exact only for linear integrands. Thus this rule does not compute the  $\mathcal{L}^2$  inner product  $(\mathbf{u}, \mathbf{v})_\tau$  exactly for linear functions (since  $\mathbf{u} \cdot \mathbf{v}$  is quadratic then). In fact, the exact integration would lead to the matrix  $D_\tau$  (three-point rule integration formula based on the midpoints of the triangle's edges). However, it provides an exact value of  $(\mathbf{u}, \bar{\mathbf{v}})_\tau$ , with  $\bar{\mathbf{v}}$  being the (piecewise constant) average value of  $\mathbf{v}$  in each triangle  $\tau$  (note that  $\|\bar{\mathbf{v}}\|_\tau^2 \leq \frac{4}{3} \|\mathbf{v}\|_\tau^2$  for  $\mathbf{v}$  linear in  $\tau$ ).

With this notation, the mini-element and Petrov-Galerkin formulations can be written as

$$(19) \quad \begin{aligned} B((\mathbf{u}_{h,l}, p_h), (\mathbf{v}, q)) &= (\mathbf{f}, \mathbf{v}) - \sum_{\tau \in \mathcal{T}} \frac{\alpha^2}{3600\sigma'_\tau} (\mathbf{u}_{h,l}, \mathbf{v} - \bar{\mathbf{v}})_\tau \\ &\quad + \sum_{\tau \in \mathcal{T}} \frac{1}{60\sigma'_\tau} (\mathbf{f}, \psi_b(\tau)(\nabla q - \alpha \bar{\mathbf{v}}))_\tau \end{aligned}$$

and

$$(20) \quad B((\mathbf{u}_L, p_L), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) + \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{f}, \nabla q - \alpha \mathbf{v})_\tau$$

for  $(\mathbf{v}, q) \in \mathcal{L}_0 \times \mathcal{L}_0 \times \mathcal{L}$ .

Subtracting (20) from (19) we get

$$\begin{aligned} B((\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L), (\mathbf{v}, q)) &= \alpha^2 \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{u}_{h,l}, \bar{\mathbf{v}} - \mathbf{v})_\tau \\ &\quad - \alpha \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{f}, \bar{\mathbf{v}} - \mathbf{v})_\tau + \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{f}, (60\psi_b(\tau) - 1)(\nabla q - \alpha \bar{\mathbf{v}}))_\tau \end{aligned}$$

**THEOREM 4.1.** *There exist two positive constants  $C_1$  and  $C_2$  depending on the minimal angle in the triangulation such that*

$$\|(\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L)\|_* \leq C_1 \frac{h^2}{\nu} \|\nabla \mathbf{f}\| + C_2 \frac{\sqrt{\alpha}}{\nu\sqrt{\nu}} h^3 (\alpha \|(\mathbf{u}_{h,l}, p_h)\|_* + \|(\mathbf{f}, 0)\|_*)$$

*Proof.* In the following  $C$  is a constant depending on the minimal angle in the triangulation  $\mathcal{T}$ . We recall the inequality  $\frac{1}{\sigma_\tau} \leq Ch_\tau^2$  for  $\tau \in \mathcal{T}$ .

We now bound each of the terms above separately; for  $(\mathbf{v}, q) \in \mathcal{L}_0 \times \mathcal{L}_0 \times \mathcal{L}$ , we get:

$$\begin{aligned} \left| \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{u}_{h,l}, \bar{\mathbf{v}} - \mathbf{v})_\tau \right| &\leq C \frac{h^2}{\nu} \|\mathbf{u}_{h,l}\| \|\bar{\mathbf{v}} - \mathbf{v}\| \\ &\leq C \frac{h^3}{\nu} \|\nabla \mathbf{v}\| \|\mathbf{u}_{h,l}\| \end{aligned}$$

Then

$$\begin{aligned} \left| \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{f}, \bar{\mathbf{v}} - \mathbf{v})_\tau \right| &\leq C \frac{h^2}{\nu} \sum_{\tau \in \mathcal{T}} \|\mathbf{f}\|_\tau \|\bar{\mathbf{v}} - \mathbf{v}\|_\tau \\ &\leq C \frac{h^3}{\nu} \|\mathbf{f}\| \|\nabla \mathbf{v}\| \end{aligned}$$

and for the third term:

$$\left| \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{f}, (60\psi_b(\tau) - 1)(\nabla q - \alpha \bar{\mathbf{v}}))_\tau \right|$$

$$\begin{aligned}
&= \left| \sum_{\tau \in \mathcal{T}} \frac{1}{3600\sigma'_\tau} (\mathbf{f} - \bar{\mathbf{f}}, (60\psi_b(\tau) - 1)(\nabla q - \alpha\bar{\mathbf{v}}))_\tau \right| \\
&\leq Ch \sum_{\tau \in \mathcal{T}} \frac{1}{\sqrt{3600\sigma'_\tau}} \|\nabla \mathbf{f}\|_\tau \frac{\|(60\psi_b(\tau) - 1)(\nabla q - \alpha\bar{\mathbf{v}})\|_\tau}{\sqrt{3600\sigma'_\tau}} \\
&\leq C \frac{h^2}{\sqrt{\nu}} \|\nabla \mathbf{f}\| \left( \left( \sum_{\tau \in \mathcal{T}} \frac{\|\nabla q\|_\tau^2}{3600\sigma'_\tau} \right)^{1/2} + C \frac{\alpha}{\sqrt{\nu}} h \|\mathbf{v}\| \right)
\end{aligned}$$

Thus, using the coercivity of  $B(\cdot, \cdot)$  over  $\mathcal{L}_0 \times \mathcal{L}_0 \times \mathcal{L}$ , we get:

$$\begin{aligned}
\beta \|(\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L)\|_*^2 &\leq B((\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L), (\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L)) \\
&\leq C \frac{\alpha^2}{\nu} h^3 \|\nabla(\mathbf{u}_{h,l} - \mathbf{u}_L)\| \|\mathbf{u}_{h,l}\| + C \frac{\alpha h^3}{\nu} \|\mathbf{f}\| \|\nabla(\mathbf{u}_{h,l} - \mathbf{u}_L)\| \\
&\quad + C \frac{h^2}{\sqrt{\nu}} \|\nabla \mathbf{f}\| \left( \left( \sum_{\tau \in \mathcal{T}} \frac{\|\nabla(p_h - p_L)\|_\tau^2}{3600\sigma'_\tau} \right)^{1/2} + C \frac{\alpha}{\sqrt{\nu}} h \|\mathbf{u}_{h,l} - \mathbf{u}_L\| \right)
\end{aligned}$$

which yields

$$\begin{aligned}
\|(\mathbf{u}_{h,l}, p_h) - (\mathbf{u}_L, p_L)\|_* &\leq C \frac{\alpha \sqrt{\alpha}}{\nu \sqrt{\nu}} h^3 \|(\mathbf{u}_{h,l}, p_h)\|_* + C \frac{\alpha}{\nu \sqrt{\nu}} h^3 \|\mathbf{f}\| \\
&\quad + C \frac{h^2}{\sqrt{\nu}} \|\nabla \mathbf{f}\| \left(1 + \frac{\sqrt{\alpha}}{\sqrt{\nu}} h\right) \\
&\leq C \frac{\sqrt{\alpha}}{\nu \sqrt{\nu}} h^3 (\alpha \|(\mathbf{u}_{h,l}, p_h)\|_* + \|(\mathbf{f}, 0)\|_*) + C \frac{h^2}{\sqrt{\nu}} \|\nabla \mathbf{f}\|
\end{aligned}$$

□

Thus the term  $\alpha \mathbf{u}$  does not create a large perturbation (order  $\mathcal{O}(h^3)$ ), when compared to the regular Stokes problem ( $\alpha = 0$ ), for which the norm of the difference between the solutions to the two formulations is of order  $\mathcal{O}(h^2)$ . However, one should remember that in the context of the full Navier Stokes equations,  $\alpha$  is of order  $\mathcal{O}(\frac{1}{\Delta t})$ . For smaller time steps the effect of the second term on the right-hand side in theorem 4.1 is no longer negligible compared to the second term. For a time step comparable to the mesh size this results in an estimate of order  $\mathcal{O}(h^{3/2})$  for example.

Both mini-element and Petrov-Galerkin solutions converge to the same value when the size of the mesh becomes small. Also, the mini-element discretization can be viewed as selecting a particular choice for triangle dependent stability constants, which must be selected in the Petrov-Galerkin scheme.

## REFERENCES

- [1] D. N. ARNOLD, F. BREZZI, AND M. FORTIN, *A stable finite element for the Stokes equations*, *Calcolo*, 21 (1984), pp. 337–344.
- [2] R. E. BANK AND B. D. WELFERT, *A posteriori error estimates for the Stokes problem*, to appear, (June 1989).
- [3] F. BREZZI AND J. J. DOUGLAS, *Stabilized mixed methods for the Stokes problem*, *Numer. Math.*, 53 (1988), pp. 225–235.
- [4] T. J. R. HUGHES, L. P. FRANCA, AND M. BALESTRA, *A new finite element formulation for computational fluid dynamics: V. circumventing the Babuška-Brezzi condition: A stable Petrov-Galerkin formulation for the Stokes problem accommodating equal-order interpolations*, *Comp. Meth. in Appl. Mech. and Engr.*, 59 (1986), pp. 85–99.

- [5] R. VERFÜRTH, *A posteriori error estimators for the Stokes equations*, Preprint Nr 445, (December 1987).