A POSTERIORI ERROR ESTIMATES BASED ON HIERARCHICAL BASES [*]

RANDOLPH E. BANK[†] AND R. KENT SMITH[‡]

This work is dedicated to the memory of Alan Weiser.

Abstract. The authors present an analysis of an a posteriori error estimator based on the use of hierarchical basis functions. The authors analyze nonlinear, nonselfadjoint and indefinite problems as well as the selfadjoint, positive-definite case. Because both the analysis and the estimator itself are quite simple, it is easy to see how various approximations affect the quality of the estimator. As examples, the authors apply the theory to some scalar elliptic equations and the Stokes system of equations.

Key words. finite element methods, adaptive mesh refinement, a posteriori error estimates, Hierarchical Basis.

AMS subject classifications. 65N50, 65N30

1. Introduction.

A posteriori error estimates are now widely used in the solution of partial differential equations [3] [7] [6]. Such estimates provide useful indications of the accuracy of a calculation and also provide the basis of adaptive local mesh refinement or local order refinement schemes, h and p refinement, respectively. In this paper, we do not introduce a new scheme, but rather analyze an existing class of algorithms in a new and we think revealing way. The scheme is based on the use of hierarchical bases of the type often used in the p version of the finite element method. One of the earliest uses of such estimators that we know of was that of Zienkiewicz et al. in the early 1980's [21] [22].

For example, if one has solved a problem for a given value of p, corresponding to a finite element space $\mathcal{M}_h$, one can enrich the space to, say, order $p+1$ by adding certain hierarchical basis functions to the set of basis functions already used for $\mathcal{M}_h$ [20]. If $\bar{\mathcal{M}}_h$ is the new space, then we have the hierarchical decomposition

$$\bar{\mathcal{M}}_h = \mathcal{M}_h \oplus \mathcal{W}_h,$$

where $\mathcal{W}_h$ is the subspace which corresponds to the span of the additional basis functions.

If we resolve the problem with the space $\bar{\mathcal{M}}_h$ using the hierarchical basis, intuitively one expects that the component of the new solution lying in $\mathcal{M}_h$ will change very little from the previous calculation. Therefore, the component lying in $\mathcal{W}_h$ should be a good approximation to the error for the solution on the original space $\mathcal{M}_h$.

In fact, for our error estimate, we simply solve an (approximate) problem on the space $\mathcal{W}_h$ rather than $\bar{\mathcal{M}}_h$ to estimate the error. Since the hierarchical basis for $\mathcal{W}_h$ is typically made up of highly oscillatory functions with compact support, one can often approximate the stiffness matrix by a diagonal matrix, which further reduces the cost of computing the error estimate.

In this paper, we prove estimates of the form

$$C_1 |||u - u_h||| \le |||e_h||| \le C_2 |||u - u_h|||,$$

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[†] Department of Mathematics, University of California at San Diego, La Jolla, CA 92093. The work of this author was supported by the Office of Naval Research under contract N00014-89J-1440.

[‡] AT&T Bell Laboratories, Murray Hill, New Jersey 07974

where u is the exact solution, $u_h \in \mathcal{M}_h$ is the approximate solution, $e_h \in \mathcal{W}_h$ is the computed a posteriori error estimate, and $|\!|\!| \cdot |\!|\!|$ is an appropriate norm. C_1 and C_2 are constants of order one. We analyze three typical situations of increasing complexity:

- The case of a selfadjoint, positive definite problem.
- The case of a problem which is not selfadjoint and positive definite, but is still linear.
- The case of a general nonlinear problem.

Some generalizations to Petrov-Galerkin methods and time dependent problems are given in [11] and [18], respectively.

The development is at a fairly abstract level, so that with proper interpretation of the bilinear forms, solutions, etc., the analysis applies to systems as well as to scalar equations. An important feature of our analysis is that the proofs are not complicated, and require only very weak assumptions about the operator. Because of this, it is easy to explicitly see how the constants C_1 and C_2 depend on the underlying assumptions.

In the selfadjoint case, presented in §2, we make a *saturation assumption* (2.7). This states that in the energy norm, the finite element solution $\bar{u}_h \in \bar{\mathcal{M}}_h$ is a better approximation to the solution u than $u_h \in \mathcal{M}_h$. Generally, one expects that increasing the order of the approximation will significantly reduce the error, so this is a very natural assumption. We also assume a *strengthened Cauchy inequality* (2.8) for the spaces $\mathcal{M}_h$ and $\mathcal{W}_h$ in the energy inner product. The existence of such an inequality is typically a direct consequence of the construction of hierarchical basis functions, and its verification is usually straightforward. Such inequalities have been widely used in the analysis of iterative methods based on hierarchical bases [8] [9] [15], but have not been used much in the present context (although see [10] and [12]).

When we consider nonselfadjoint problems in §3, we add a *continuity assumption* (3.2) and an *inf-sup* condition (3.3) to our list of assumptions. These are standard assumptions made for such problems, and must be satisfied in order for the finite element formulation to be well defined [2].

In the case of nonlinear problems analyzed in §4, we assume that near the solution, some linearization of the problem satisfies the four assumptions mentioned above, and that the terms neglected in the linearization process are not too large (4.7)-(4.8). It is difficult to state what the standard assumptions for this type of problem should be, but we think the ones made here are quite reasonable. In this case, our error estimate is computed by solving a *linear* problem in order to reduce computation costs.

In §§2-3 we also analyze the effect of approximating the stiffness matrix by a matrix which is more easily inverted (e.g., the diagonal of the stiffness matrix). For the selfadjoint case, we assume the two matrices are comparable, while for the nonselfadjoint case, we assume continuity and inf-sup conditions.

In §5, we give some illustrative examples for scalar equations and for the Stokes system of equations. We don't present any numerical experiments, but this and related methods are widely used, and many numerical illustrations already exist in the literature [6] [21] [22] [7] [12].

2. The selfadjoint case. In this section, we consider the solution of the selfadjoint variational equation: find $u \in \mathcal{H}$ such that

$$a(u, v) = f(v) \tag{2.1}$$

for all $v \in \mathcal{H}$, where $\mathcal{H}$ is an appropriate Hilbert space, $a(\cdot, \cdot)$ is a positive definite bilinear form, and $f(\cdot)$ is a linear functional. The energy norm associated with $a(\cdot, \cdot)$

is denoted by

$$\|u\|^2 = a(u,u) \tag{2.2}$$

Let $\mathcal{M}_h \subset \mathcal{H}$ be a member of a family of finite dimensional subspaces, characterized by a small parameter h, and consider the approximate problem: find $u_h \in \mathcal{M}_h$ such that

$$a(u_h, v) = f(v) \tag{2.3}$$

for all $v \in \mathcal{M}_h$. The solution of (2.3) satisfies the *best approximation* property

$$\|u - u_h\| = \inf_{v \in \mathcal{M}_h} \|u - v\| \tag{2.4}$$

We will assume some notion of convergence; that is

$$\|u - u_h\| \to 0$$

as $h \to 0$.

We now define a larger space $\mathcal{M}_h \subset \bar{\mathcal{M}}_h \subset \mathcal{H}$. With this space we have an approximate solution $\bar{u}_h$ satisfying

$$a(\bar{u}_h, v) = f(v) \tag{2.5}$$

for all $v \in \bar{\mathcal{M}}_h$, and

$$\|u - \bar{u}_h\| = \inf_{v \in \bar{\mathcal{M}}_h} \|u - v\| \tag{2.6}$$

Although we don't explicitly compute $\bar{u}_h$, it enters into our theoretical analysis of the a posteriori error estimate for u_h. In particular, we assume that the approximate solutions $\bar{u}_h$ converge to u more rapidly than u_h. This is expressed in terms of the *saturation assumption*

$$\|u - \bar{u}_h\| \leq \beta \, \|u - u_h\|, \tag{2.7}$$

where $\beta < 1$ independent of h. (We note that since $\mathcal{M}_h \subset \bar{\mathcal{M}}_h$, $\beta \leq 1$ is insured by the best approximation property.) In a typical situation, due to the higher degree of approximation for the space $\bar{\mathcal{M}}_h$, one can anticipate that $\beta = O(h^r)$, for some $r > 0$. In this case, $\beta \to 0$ as $h \to 0$, which is stronger than required by our theorems.

We assume that the space $\bar{\mathcal{M}}_h$ has a hierarchical decomposition

$$\bar{\mathcal{M}}_h = \mathcal{M}_h \oplus \mathcal{W}_h.$$

Then any function $z \in \bar{\mathcal{M}}_h$ has the unique decomposition $z = v + w$, where $v \in \mathcal{M}_h$ and $w \in \mathcal{W}_h$. Additionally, we assume a *strengthened Cauchy inequality* for the decomposition; that is for all $v \in \mathcal{M}_h$ and $w \in \mathcal{W}_h$,

$$|a(v,w)| \leq \gamma \, \|v\| \, \|w\|, \tag{2.8}$$

where $\gamma < 1$ independent of h.

We seek to approximate the error $u - u_h$ in the space $\mathcal{W}_h$. Our a posteriori error estimate is defined by: find $e_h \in \mathcal{W}_h$ such that

$$a(e_h, v) = f(v) - a(u_h, v) \tag{2.9}$$

for all $v \in \mathcal{W}_h$.

We note the orthogonality relations

$$a(u - u_h, v) = 0 \qquad \text{for all } v \in \mathcal{M}_h \tag{2.10}$$

$$a(u - \bar{u}_h, v) = 0 \qquad \text{for all } v \in \bar{\mathcal{M}}_h \tag{2.11}$$

$$a(\bar{u}_h - u_h, v) = 0 \qquad \text{for all } v \in \mathcal{M}_h \tag{2.12}$$

$$a(u - u_h - e_h, v) = 0 \qquad \text{for all } v \in \mathcal{W}_h \tag{2.13}$$

$$a(\bar{u}_h - u_h - e_h, v) = 0 \qquad \text{for all } v \in \mathcal{W}_h \tag{2.14}$$

Equations (2.10)-(2.14) are proved using various combinations of (2.1), (2.3), (2.5), and (2.9), restricted to the indicated subspaces. We can use the orthogonality relationships (2.10)-(2.12) to show

$$|||u - u_h|||^2 = |||u - \bar{u}_h|||^2 + |||\bar{u}_h - u_h|||^2. \tag{2.15}$$

Using (2.15) in conjunction with the saturation assumption (2.7) shows

$$(1 - \beta^2)|||u - u_h|||^2 \leq |||\bar{u}_h - u_h|||^2 \leq |||u - u_h|||^2, \tag{2.16}$$

demonstrating $\bar{u}_h - u_h$ to be a good approximation to the error. However, our goal is to show the easily computed function e_h also yields a good approximation of the error. This is shown in

Theorem 2.1. *Let $\bar{\mathcal{M}}_h = \mathcal{M}_h \oplus \mathcal{W}_h$ as above and assume (2.7) and (2.8) hold. Then*

$$(1 - \beta^2)(1 - \gamma^2)\, |||u - u_h|||^2 \leq |||e_h|||^2 \leq |||u - u_h|||^2. \tag{2.17}$$

Proof. The right inequality in (2.17) is a simple consequence of (2.13) for the choice $v = e_h$. Now let $\bar{u}_h = \hat{u}_h + \hat{e}_h$, where $\hat{u}_h \in \mathcal{M}_h$, and $\hat{e}_h \in \mathcal{W}_h$. Then, using (2.12) with $v = \hat{u}_h - u_h$ and (2.14) with $v = \hat{e}_h$, we obtain

$$\begin{aligned} |||\bar{u}_h - u_h|||^2 &= a(\bar{u}_h - u_h, \hat{e}_h) \\ &= a(e_h, \hat{e}_h). \end{aligned} \tag{2.18}$$

Combining this with (2.15), we get

$$|||u - u_h|||^2 = |||u - \bar{u}_h|||^2 + a(\hat{e}_h, e_h). \tag{2.19}$$

To complete the proof, we must estimate $|||\hat{e}_h|||$ in terms of $|||e_h|||$. We apply the strengthened Cauchy inequality (2.8) to obtain

$$\begin{aligned} |||\bar{u}_h - u_h|||^2 &\geq |||\hat{u}_h - u_h|||^2 + |||\hat{e}_h|||^2 - 2\gamma\, |||\hat{u}_h - u_h|||\ |||\hat{e}_h||| \\ &\geq (1 - \gamma^2)|||\hat{e}_h|||^2. \end{aligned} \tag{2.20}$$

Combine this with (2.18) to obtain

$$(1 - \gamma^2)|||\hat{e}_h||| \leq |||e_h|||. \tag{2.21}$$

Using(2.19) and (2.21), we have

$$|||u - u_h|||^2 \leq \beta^2 |||u - u_h|||^2 + \frac{1}{1 - \gamma^2}|||e_h|||^2.$$

Rearranging this inequality leads directly to the left-hand inequality in (2.17). ▯

We next consider the effect of approximating $a(\cdot,\cdot)$ on the left hand side of (2.9), by a bilinear form which is more easily inverted. Thus we are lead to a modified process in which (2.9) is replaced by: find $\tilde{e}_h \in \mathcal{W}_h$ such that

$$b(\tilde{e}_h, v) = f(v) - a(u_h, v) \tag{2.22}$$

for all $v \in \mathcal{W}_h$.

Theorem 2.2. *Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be symmetric, positive definite bilinear forms, and let c_0 and c_1 be positive constants such that*

$$c_0 \le \frac{b(w,w)}{a(w,w)} \le c_1 \tag{2.23}$$

for all $w \in \mathcal{W}_h$, $w \neq 0$. Let e_h and $\tilde{e}_h$ be defined by (2.9) and (2.22), respectively. Then

$$c_0 \, |||\tilde{e}_h||| \le |||e_h||| \le c_1 \, |||\tilde{e}_h|||. \tag{2.24}$$

Proof. From (2.9) and (2.22) we obtain the relation

$$b(\tilde{e}_h, v) = a(e_h, v)$$

for all $v \in \mathcal{W}_h$. Taking $v = \tilde{e}_h$ and using (2.23), we have

$$\begin{aligned} c_0 |||\tilde{e}_h|||^2 &\le b(\tilde{e}_h, \tilde{e}_h) \\ &= a(e_h, \tilde{e}_h) \\ &\le |||e_h|||\,|||\tilde{e}_h||| \end{aligned}$$

Similarly, taking $v = e_h$ leads to

$$\begin{aligned} |||e_h|||^2 &= a(e_h, e_h) \\ &= b(\tilde{e}_h, e_h) \\ &\le c_1 |||e_h|||\,|||\tilde{e}_h|||. \end{aligned}$$

▯

Combining Theorems 2.1 and 2.2, we obtain the final estimate

$$c_1^{-2}(1-\beta^2)(1-\gamma^2)\, |||u-u_h|||^2 \le |||\tilde{e}_h|||^2 \le c_0^{-2} |||u-u_h|||^2. \tag{2.25}$$

3. The nonselfadjoint, indefinite case. In this section, we generalize the results of §2 to the nonselfadjoint and possibly indefinite problem: find $u \in \mathcal{H}$ such that

$$A(u, v) = f(v) \tag{3.1}$$

for all $v \in \mathcal{H}$, where $A(\cdot,\cdot)$ is a bilinear form and $f(\cdot)$ is a linear functional. The energy norm $|||\cdot|||$ is associated with the positive definite bilinear form $a(\cdot,\cdot)$ defined in §2. The finite-dimensional spaces $\mathcal{M}_h$, $\bar{\mathcal{M}}_h$, and $\mathcal{W}_h$ are defined as in §2. We assume the bilinear form $A(\cdot,\cdot)$ satisfies the *continuity condition*

$$|A(\phi, \eta)| \le \nu \, |||\phi||| \; |||\eta||| \tag{3.2}$$

for all $\phi, \eta \in \mathcal{H}$, and the *inf-sup condition*

$$\inf_{\substack{\phi \in \mathcal{S} \\ |||\phi||| = 1}} \sup_{\substack{\eta \in \mathcal{S} \\ |||\eta||| \leq 1}} A(\phi, \eta) \geq \mu > 0, \tag{3.3}$$

where $\mathcal{S} = \mathcal{H}, \mathcal{M}_h, \bar{\mathcal{M}}_h, \mathcal{W}_h$. This insures that all the systems of interest will have unique solutions.

The approximate solutions $u_h \in \mathcal{M}_h$, $\bar{u}_h \in \bar{\mathcal{M}}_h$, and the a posteriori error estimate $e_h \in \mathcal{W}_h$ satisfy

$$A(u_h, v) = f(v) \qquad \text{for all } v \in \mathcal{M}_h \tag{3.4}$$

$$A(\bar{u}_h, v) = f(v) \qquad \text{for all } v \in \bar{\mathcal{M}}_h \tag{3.5}$$

$$A(u_h, v) + A(e_h, v) = f(v) \qquad \text{for all } v \in \mathcal{W}_h. \tag{3.6}$$

As in §2, we assume that the solutions u_h and $\bar{u}_h$ both converge to u and that the saturation assumption (2.7) holds. We also assume the strengthened Cauchy inequality (2.8) for the inner product $a(\cdot, \cdot)$.

We note the relations

$$A(u - u_h, v) = 0 \qquad \text{for all } v \in \mathcal{M}_h \tag{3.7}$$

$$A(u - \bar{u}_h, v) = 0 \qquad \text{for all } v \in \bar{\mathcal{M}}_h \tag{3.8}$$

$$A(\bar{u}_h - u_h, v) = 0 \qquad \text{for all } v \in \mathcal{M}_h \tag{3.9}$$

$$A(u - u_h - e_h, v) = 0 \qquad \text{for all } v \in \mathcal{W}_h \tag{3.10}$$

$$A(\bar{u}_h - u_h - e_h, v) = 0 \qquad \text{for all } v \in \mathcal{W}_h \tag{3.11}$$

in analogy to (2.10)-(2.14).

Using the triangle inequality with the saturation assumption (2.7) shows

$$(1 - \beta)|||u - u_h||| \leq |||\bar{u}_h - u_h||| \leq (1 + \beta)|||u - u_h||| \tag{3.12}$$

in analogy to (2.16).

THEOREM 3.1. *Let $\bar{\mathcal{M}}_h = \mathcal{M}_h \oplus \mathcal{W}_h$ as above, and assume (3.2), (3.3), (2.7), and (2.8) hold. Then for h sufficiently small,*

$$\left(\frac{\mu}{\nu}\right)^2 (1 - \beta)^2 (1 - \gamma^2)\, |||u - u_h|||^2 \leq |||e_h|||^2 \leq \left(\frac{\nu}{\mu}\right)^2 |||u - u_h|||^2. \tag{3.13}$$

Proof. First let $w \in \mathcal{W}_h$. Using (3.10), (3.2), and (3.3), we have

$$\begin{aligned} \mu\, |||e_h||| &\leq \sup_{|||w|||=1} A(e_h, w) \\ &= \sup_{|||w|||=1} A(u - u_h, w) \\ &\leq \nu\, |||u - u_h|||, \end{aligned}$$

proving the right-hand inequality in (3.13).

Now let $v \in \mathcal{M}_h$, $w \in \mathcal{W}_h$, with $|||v + w||| = 1$; then

$$\begin{aligned} 1 &= |||v + w|||^2 \\ &= |||v|||^2 + |||w|||^2 + 2a(v, w) \\ &\geq |||v|||^2 + |||w|||^2 - 2\gamma\, |||v|||\ |||w||| \\ &\geq (1 - \gamma^2)\, |||w|||^2. \end{aligned}$$

Thus, using (3.9), (3.11), (3.3), and (3.2)

$$\begin{aligned}
\mu \,\|\bar{u}_h - u_h\| &\le \sup_{\|v+w\|=1} A(\bar{u}_h - u_h, v+w) \\
&= \sup_{\|v+w\|=1} A(\bar{u}_h - u_h, w) \\
&= \sup_{\|v+w\|=1} A(e_h, w) \\
&\le \frac{\nu}{\sqrt{1-\gamma^2}} \,\|e_h\|.
\end{aligned}$$

This in conjunction with (3.12) establishes the left inequality in (3.13), provided h is sufficiently small. ☐

We next analyze the effect of approximating $A(\cdot,\cdot)$ by the (more easily inverted) bilinear form $B(\cdot,\cdot)$ in the computation of the a posteriori error estimate. We define $\tilde{e}_h \in \mathcal{W}_h$ by

$$A(u_h, v) + B(\tilde{e}_h, v) = f(v) \tag{3.14}$$

for all $v \in \mathcal{W}_h$. We assume that $B(\cdot,\cdot)$ satisfies the continuity condition

$$|B(\phi,\eta)| \le \tilde{\nu} \,\|\phi\|\, \|\eta\| \tag{3.15}$$

for all $\phi, \eta \in \mathcal{W}_h$, and the inf-sup condition

$$\inf_{\substack{\phi \in \mathcal{W}_h \\ \|\phi\| = 1}} \sup_{\substack{\eta \in \mathcal{W}_h \\ \|\eta\| \le 1}} B(\phi,\eta) \ge \tilde{\mu} > 0. \tag{3.16}$$

THEOREM 3.2. *Let $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ be bilinear forms satisfying (3.2)-(3.3) and (3.15)-(3.16) respectively. Let e_h and $\tilde{e}_h$ be defined by (3.6) and (3.14). Then*

$$\frac{\tilde{\mu}}{\nu} \,\|\tilde{e}_h\| \le \|e_h\| \le \frac{\tilde{\nu}}{\mu} \,\|\tilde{e}_h\|. \tag{3.17}$$

Proof. From (3.6) and (3.14) we obtain the relation

$$B(\tilde{e}_h, v) = A(e_h, v)$$

for all $v \in \mathcal{W}_h$. The inequalities

$$\begin{aligned}
\tilde{\mu} \,\|\tilde{e}_h\| &\le \nu \,\|e_h\| \\
\mu \,\|e_h\| &\le \tilde{\nu} \,\|\tilde{e}_h\|
\end{aligned}$$

follow immediately from the inf-sup and continuity conditions. ☐

4. The nonlinear case. In this section we generalize the results of §§2-3 to the nonlinear problem: find $u \in \mathcal{H}$ such that

$$\mathcal{A}(u, v) = f(v) \tag{4.1}$$

for all $v \in \mathcal{H}$, where $\mathcal{A}(u,v)$ is nonlinear in u but still linear in v, and $f(\cdot)$ is a linear functional. The finite-dimensional spaces $\mathcal{M}_h$, $\bar{\mathcal{M}}_h$, and $\mathcal{W}_h$ are defined as in §2. The bilinear forms $a(\cdot,\cdot)$ and $A(\cdot,\cdot)$ and the energy norm $\|\cdot\|$ are as defined in §§2-3.

The approximate solutions $u_h \in \mathcal{M}_h$, $\bar{u}_h \in \bar{\mathcal{M}}_h$, and the a posteriori error estimate $e_h \in \mathcal{W}_h$ satisfy

$$\mathcal{A}(u_h, v) = f(v) \qquad \text{for all } v \in \mathcal{M}_h \tag{4.2}$$

$$\mathcal{A}(\bar{u}_h, v) = f(v) \qquad \text{for all } v \in \bar{\mathcal{M}}_h \tag{4.3}$$

$$\mathcal{A}(u_h, v) + A(e_h, v) = f(v) \qquad \text{for all } v \in \mathcal{W}_h. \tag{4.4}$$

The bilinear form $A(\cdot,\cdot)$ is assumed to be related to $\mathcal{A}(\cdot,\cdot)$ through some linearization process. In particular, $A(\cdot,\cdot)$ could correspond to the Jacobian of $\mathcal{A}(\cdot,\cdot)$ evaluated at the discrete solution u_h. Note that computing the error estimate e_h requires the solution of a *linear* problem. We can replace the bilinear form $A(\cdot,\cdot)$ in (4.4) with $B(\cdot,\cdot)$, as in §3. The effect of this substitution is quantified in Theorem 3.2 and thus is not considered here.

To quantify the effect of the linearization process, and to make explicit the relationship between the forms $\mathcal{A}(\cdot,\cdot)$ and $A(\cdot,\cdot)$, we introduce the form $Q(w - u_h, v)$ defined by

$$Q(w - u_h, v) = \mathcal{A}(w, v) - \mathcal{A}(u_h, v) - A(w - u_h, v).$$

From (4.1)-(4.3) we have the relations

$$\mathcal{A}(u, v) - \mathcal{A}(u_h, v) = A(u - u_h, v) + Q(u - u_h, v) = 0 \tag{4.5}$$

$$\mathcal{A}(\bar{u}_h, v) - \mathcal{A}(u_h, v) = A(\bar{u}_h - u_h, v) + Q(\bar{u}_h - u_h, v) = 0 \tag{4.6}$$

for all $v \in \mathcal{M}_h$. We assume that

$$|Q(u - u_h, v)| \le \delta \, |||u - u_h||| \, |||v||| \tag{4.7}$$

$$|Q(\bar{u}_h - u_h, v)| \le \delta \, |||\bar{u}_h - u_h||| \, |||v||| \tag{4.8}$$

for all $v \in \bar{\mathcal{M}}_h$, and $\delta \to 0$ as $h \to 0$. Normally, we anticipate that $Q(\cdot,\cdot)$ will correspond to the (truncated) quadratic terms in the linearization process (e.g., $Q(u - u_h, v) = O(|||u - u_h|||^2 \, |||v|||)$), and this provides the motivation for the assumptions (4.7)-(4.8). Also note that $Q(w - u_h, v)$ is a linear functional with respect to v.

Using (4.1)-(4.4), we obtain the relations

$$\mathcal{A}(u, v) - \mathcal{A}(\bar{u}_h, v) = A(u - \bar{u}_h, v) + Q(u - u_h, v) - Q(\bar{u}_h - u_h, v) = 0 \tag{4.9}$$

for all $v \in \bar{\mathcal{M}}_h$, and

$$\mathcal{A}(u, v) - \mathcal{A}(u_h, v) - A(e_h, v) = A(u - u_h - e_h, v) + Q(u - u_h, v) = 0 \tag{4.10}$$

$$\mathcal{A}(\bar{u}_h, v) - \mathcal{A}(u_h, v) - A(e_h, v) = A(\bar{u}_h - u_h - e_h, v) + Q(\bar{u}_h - u_h, v) = 0 \tag{4.11}$$

for all $v \in \mathcal{W}_h$.

We continue to make the main assumptions used in §§2-3, namely (3.2), (3.3), (2.7), and (2.8). In particular, since now $A(\cdot,\cdot)$ depends on u_h, we assume that the constants μ and ν are uniform in some ball about the true solution u, large enough to contain all approximate solutions of interest.

THEOREM 4.1. *Let $\bar{\mathcal{M}}_h = \mathcal{M}_h \oplus \mathcal{W}_h$ as above, and assume (3.2), (3.3), (2.7), (2.8), and (4.7)-(4.8) hold. Then, for h sufficiently small,*

$$\left(\frac{\mu - \delta}{\nu}\right)^2 (1 - \gamma^2)(1 - \beta)^2 \, |||u - u_h|||^2 \le |||e_h|||^2 \le \left(\frac{\nu + \delta}{\mu}\right)^2 |||u - u_h|||^2. \tag{4.12}$$

Proof. First let $w \in \mathcal{W}_h$. Using (4.7) and (4.10) we have

$$\begin{aligned}
\mu \, |||e_h||| &\le \sup_{|||w|||=1} A(e_h, w) \\
&= \sup_{|||w|||=1} A(u - u_h, w) + Q(u - u_h, w) \\
&\le (\nu + \delta) \, |||u - u_h|||,
\end{aligned}$$

proving the second inequality in (4.12).

Now let $v \in \mathcal{M}_h$, $w \in \mathcal{W}_h$, with $|||v + w||| = 1$. Then $\sqrt{1-\gamma^2}\, |||w||| \le 1$. Using (4.6), (4.8), and (4.11), we have

$$\begin{aligned}
\mu \, |||\bar{u}_h - u_h||| &\le \sup_{|||v+w|||=1} A(\bar{u}_h - u_h, v + w) \\
&= \sup_{|||v+w|||=1} A(\bar{u}_h - u_h, w) - Q(\bar{u}_h - u_h, v) \\
&= \sup_{|||v+w|||=1} A(e_h, w) - Q(\bar{u}_h - u_h, v + w) \\
&\le \frac{\nu}{\sqrt{1-\gamma^2}} \, |||e_h||| + \delta \, |||\bar{u}_h - u_h|||.
\end{aligned}$$

Now, provided h is sufficiently small, this, in conjunction with (3.12), establishes the first inequality in (4.12). □

5. Examples. In this section, we present a few examples that illustrate the use of the error estimators developed in §§2-4.

5.1. Example 1. As our first example, we consider the solution of the Poisson equation

$$-\Delta u = f \tag{5.1}$$

in $x \in \Omega \subset \mathcal{R}^2$, with boundary conditions

$$u = 0 \tag{5.2}$$

for $x \in \partial\Omega$. For simplicity, we will assume that Ω is a polygonal region. The weak form of (5.1)-(5.2) is: find $u \in \mathcal{H}_0^1(\Omega)$ such that

$$a(u, v) = f(v) \tag{5.3}$$

for all $v \in \mathcal{H}_0^1(\Omega)$, where

$$\begin{aligned}
a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \\
f(v) &= \int_\Omega f v \, dx,
\end{aligned}$$

and $\mathcal{H}_0^1(\Omega)$ is the usual subspace of the Sobolev space $\mathcal{H}^1(\Omega)$ whose elements satisfy the homogeneous boundary conditions. The energy norm $|||\cdot|||$ is given by (2.2).

Let $\mathcal{T}_h$ denote a shape regular, but not necessarily quasi uniform, triangulation of Ω, characterized by a small parameter h. The finite element space $\mathcal{M}_h \subset \mathcal{H}_0^1(\Omega)$ is the space of continuous piecewise linear polynomials associated with $\mathcal{T}_h$. It is characterized in terms of the standard nodal (Lagrange) basis. The space $\bar{\mathcal{M}}_h \subset \mathcal{H}_0^1(\Omega)$ will be the space of continuous piecewise quadratic polynomials associated with $\mathcal{T}_h$. The hierarchical basis for $\bar{\mathcal{M}}_h$ will be composed of the piecewise linear nodal basis functions associated with the vertices of $\mathcal{T}_h$, and the piecewise quadratic nodal basis functions associated with the edge midpoints of $\mathcal{T}_h$ (the so-called bump functions). With this basis, the subspace $\mathcal{W}_h$ is just the span of the bump functions, that is, continuous piecewise quadratic polynomials which are zero at the vertices of $\mathcal{T}_h$. This is a standard hierarchical decomposition, and it is well known [16] [8] that for all $v \in \mathcal{M}_h$ and all $w \in \mathcal{W}_h$,

$$|a(v,w)| \leq \gamma |\!|\!| v |\!|\!| \, |\!|\!| w |\!|\!|,$$

where $\gamma < 1$ depends only on the shape regularity of the triangles in $\mathcal{T}_h$.

Standard a priori estimates show that the finite element solution $u_h \in \mathcal{M}_h$ satisfies (2.4) and

$$|\!|\!| u - u_h |\!|\!| \leq Ch \, \|u\|_{\mathcal{H}^2(\Omega)}, \tag{5.4}$$

provided $u \in \mathcal{H}^2(\Omega)$. The finite element solution $\bar{u}_h \in \bar{\mathcal{M}}_h$ satisfies (2.6) and

$$|\!|\!| u - \bar{u}_h |\!|\!| \leq \bar{C}h^2 \, \|u\|_{\mathcal{H}^3(\Omega)}, \tag{5.5}$$

provided $u \in \mathcal{H}^3(\Omega)$. Thus, we should expect (but not require) that $\beta = O(h)$ in (2.7).

Since (2.7) and (2.8) are satisfied, Theorem 2.1 will be satisfied by the approximate error $e_h \in \mathcal{W}_h$ given in (2.9).

It is well known [8] that the stiffness matrix corresponding to (2.9), when assembled using the hierarchical basis, is uniformly comparable to its diagonal in the sense of (2.23), where c_0 and c_1 depend only on the shape regularity of the triangles in $\mathcal{T}_h$. Therefore, we can apply Theorem 2.2 in the case where $b(\cdot,\cdot)$ in (2.22) corresponds to the diagonal of the stiffness matrix and obtain the bound (2.25) for the resulting error estimate $\tilde{e}_h \in \mathcal{W}_h$.

5.2. Example 2. Our second example is the convection-diffusion equation

$$-\Delta u + \boldsymbol{\omega} \cdot \nabla u = f \tag{5.6}$$

for $x \in \Omega$, with boundary condition given by (5.2). The region Ω, the bilinear form $a(\cdot,\cdot)$, the energy norm $|\!|\!| \cdot |\!|\!|$, and the finite element spaces $\mathcal{M}_h$, $\bar{\mathcal{M}}_h$, and $\mathcal{W}_h$ are all as in the first example. We assume that the convection velocity $\boldsymbol{\omega}$ is constant and small enough that the standard Galerkin approximation is appropriate (see [11] for the case of Petrov-Galerkin approximations). The bilinear form $A(\cdot,\cdot)$ is given by

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v + \boldsymbol{\omega} \cdot \nabla u v \, dx \tag{5.7}$$

and

$$f(v) = \int_\Omega f v \, dx$$

as in Example 1.

It is easy to check that the inf-sup condition (3.3) holds with $\mu = 1$ and the continuity condition (3.2) holds with $\nu = 1 + c|\boldsymbol{\omega}|$. A priori estimates based on the arguments of Schatz [19] show (5.4) and (5.5) hold for the finite element solutions $u_h \in \mathcal{M}_h$ and $\bar{u}_h \in \bar{\mathcal{M}}_h$, respectively. Again, we can suppose that $\beta = O(h)$ as $h \to 0$. The constant γ is the same as in Example 1. Thus, we can apply Theorem 3.1 and obtain the estimate (3.13) for the approximate error $e_h \in \mathcal{W}_h$ given by (3.6). We can approximate the stiffness matrix for the bump functions by its diagonal as in the first example. The constants $\tilde{\mu}$ and $\tilde{\nu}$ in (3.15) and (3.16), respectively, will depend on $|\boldsymbol{\omega}|$ as well as the shape regularity of the elements.

5.3. Example 3. For our next example, we consider the solution of the mildly nonlinear equation

$$-\Delta u + f(u) = 0 \tag{5.8}$$

in $x \in \Omega \subset \mathcal{R}^2$, with boundary conditions given by (5.2). We assume that the nonlinear function $f(u)$ is smooth, satisfying the uniform bounds

$$0 \le \frac{\partial f}{\partial u} \le M \tag{5.9}$$

and

$$\left| \frac{\partial f(w)}{\partial u} - \frac{\partial f(v)}{\partial u} \right| \le L|w - v| \tag{5.10}$$

The region Ω, the bilinear form $a(\cdot,\cdot)$, the energy norm $|\!|\!| \cdot |\!|\!|$, and the finite element spaces $\mathcal{M}_h$, $\bar{\mathcal{M}}_h$, and $\mathcal{W}_h$ are all as in the first example. The weak formulation of (5.8) is given by: find $u \in \mathcal{H}_0^1(\Omega)$ such that

$$\mathcal{A}(u, v) = \int_\Omega \nabla u \cdot \nabla v + f(u) v \, dx = 0 \tag{5.11}$$

for all $v \in \mathcal{H}_0^1(\Omega)$. The bilinear form $A(\cdot,\cdot)$ is given by

$$A(w, v) = \int_\Omega \nabla w \cdot \nabla v + \frac{\partial f(u_h)}{\partial u} w v \, dx. \tag{5.12}$$

$A(\cdot,\cdot)$ satisfies the inf-sup condition (3.3) with $\mu = 1$ and the continuity condition (3.2) with $\nu \le 1 + cM$.

The form $Q(w - u_h, v)$ is defined by

$$\begin{aligned} Q(w - u_h, v) &= \mathcal{A}(w, v) - \mathcal{A}(u_h, v) - A(w - u_h, v) \\ &= \int_\Omega \left\{ f(w) - f(u_h) - \frac{\partial f(u_h)}{\partial u}(w - u_h) \right\} v \, dx \\ &= \int_\Omega \left\{ \frac{f(w) - f(u_h)}{w - u_h} - \frac{\partial f(u_h)}{\partial u} \right\} (w - u_h) v \, dx \end{aligned} \tag{5.13}$$

and satisfies (4.7)-(4.8) with $\delta = O(\|u - u_h\|_{\mathcal{L}^\infty(\Omega)})$ as $h \to 0$. Thus the error estimate e_h given by (4.4) will satisfy the bounds (4.12) given in Theorem 4.1.

5.4. Example 4. As our last example, we consider a simple elliptic system, the Stokes equations

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0 \tag{5.14}$$

in $x \in \Omega \subset \mathcal{R}^2$, with homogeneous boundary conditions given by (5.2). Now $\boldsymbol{u} = (u_1, u_2)^t$ is a vector velocity field, and p is the pressure. The pressure is determined only up to an additive constant.

We will compute a finite element approximation using the mini-element discretization of Arnold, Brezzi, and Fortin [1]. This method uses the bilinear form

$$A(\{\boldsymbol{u}, p\}, \{\boldsymbol{v}, q\}) = \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} - \nabla \cdot \boldsymbol{v}\, p - \nabla \cdot \boldsymbol{u}\, q\, dx, \tag{5.15}$$

where

$$\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} = \sum_{i=1}^{2} \nabla u_i \cdot \nabla v_i,$$

and the linear functional

$$f(\boldsymbol{v}, q) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

The energy inner product is

$$a(\{\boldsymbol{u}, p\}, \{\boldsymbol{v}, q\}) = \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + pq\, dx, \tag{5.16}$$

giving rise to the energy norm

$$|\!|\!| \boldsymbol{u}, p |\!|\!|^2 = \|\nabla \boldsymbol{u}\|^2_{\mathcal{L}^2(\Omega)} + \|p\|^2_{\mathcal{L}^2(\Omega)}. \tag{5.17}$$

The triangulation $\mathcal{T}_h$ will be as in the first three examples. The space $\mathcal{M}_h$ is the usual mini-element space. The velocity components are approximated using continuous piecewise linear polynomials satisfying the Dirichlet boundary conditions, plus the cubic *bubble* functions associated with the barycenter of each element. The pressure is approximated by a continuous piecewise linear polynomial. The pressure can be made unique by requiring it to have average value zero. This requirement can be easily imposed as part of the solution process and does not affect the computational basis, which is just the span of the usual nodal basis functions. The space $\bar{\mathcal{M}}_h$ is the second member of the family of mini-element spaces [1]. Each velocity component is approximated using continuous piecewise quadratic polynomials plus the quartic bubble functions. The pressure is continuous piecewise quadratic.

This pair of mini-element spaces is nested. The cubic bubble for a given triangle can be expressed as a simple linear combination of the three quartic bubbles for the same triangle. Normally, equations for the bubble functions are statically condensed from the system of linear equations to be solved, so that the unknowns that are actually computed correspond to the degrees of freedom associated with the linear and quadratic basis functions only. Thus, we define $\mathcal{M}_h = \mathcal{M}_h^\dagger \oplus \mathcal{B}_h$, where $\mathcal{M}_h^\dagger$ are just the piecewise linear functions and $\mathcal{B}_h$ are the cubic bubbles functions. Similarly, we have $\bar{\mathcal{M}}_h = \bar{\mathcal{M}}_h^\dagger \oplus \bar{\mathcal{B}}_h$, where $\bar{\mathcal{M}}_h^\dagger$ are the piecewise quadratic polynomials and $\bar{\mathcal{B}}_h$ are the quartic bubble functions. Note $\mathcal{B}_h \subset \bar{\mathcal{B}}_h$. Now we have the hierarchical decomposition $\bar{\mathcal{M}}_h = \mathcal{M}_h^\dagger \oplus \mathcal{W}_h^\dagger \oplus \bar{\mathcal{B}}_h$, where $\mathcal{W}_h^\dagger$ is the space of quadratic bump functions as in the other examples. We will take $\mathcal{W}_h = \mathcal{W}_h^\dagger \oplus \bar{\mathcal{B}}_h$.

We now verify the hypotheses for Theorem 3.1. The continuity condition (3.2) is straightforward to check. The inf-sup condition (3.3) for the spaces $\mathcal{H} \equiv \mathcal{H}_0^1(\Omega) \times \mathcal{H}_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$, $\mathcal{M}_h$ and $\bar{\mathcal{M}}_h$ are standard results [1]. (Note that $\mathcal{L}_0^2(\Omega)$ is the subspace of $\mathcal{L}^2(\Omega)$ whose elements have average value zero). Thus, the solutions $\{\boldsymbol{u}_h, p_h\}$ and $\{\bar{\boldsymbol{u}}_h, \bar{p}_h\}$ satisfy the saturation assumption with $\beta = O(h)$, provided $\{\boldsymbol{u}, p\}$ is sufficiently smooth.

To prove the inf-sup condition for $\mathcal{W}_h$, one can use a variation of the argument used in [1] (§2) for the mini-element spaces themselves; one lets the space $\bar{\mathcal{M}}_h$ play the role analogous to $\mathcal{H}$ and $\mathcal{W}_h$ play the role analogous to $\bar{\mathcal{M}}_h$. The argument simplifies somewhat because both spaces are finite dimensional, and one can use strengthened Cauchy inequalities to bound the norm of the interpolation operator.

A slightly tricky technical point in the analysis concerns the strengthened Cauchy inequality (2.8). Because $\mathcal{B}_h \subset \bar{\mathcal{B}}_h$, we must use the hierarchical decomposition $\bar{\mathcal{M}}_h = \mathcal{M}_h^\dagger \oplus \mathcal{W}_h$. Let $\{\boldsymbol{v}, q\} \in \mathcal{M}_h^\dagger$, $\{\boldsymbol{w}, r\} \in \mathcal{W}_h$. Then the relevant strengthened Cauchy inequality is

$$|a(\{\boldsymbol{v}, q\}, \{\boldsymbol{w}, r\})| \leq \gamma \, |\!|\!| \boldsymbol{v}, q |\!|\!| \; |\!|\!| \boldsymbol{w}, r |\!|\!|,$$

which is established in the usual fashion. One can check that the argument used in the proof of Theorem 3.1 is affected in only a trivial way by this modification.

Now, since all the hypothesis are satisfied, we can apply a slightly modified version of Theorem 3.1 to the hierarchical basis error estimate for the mini-element discretization.

6. Comments and concluding remarks. As one can see from the foregoing examples, hierarchical basis a posteriori error estimators can be applied to a wide variety of partial differential equations. In situations where it might be inconvenient or impossible to directly verify the hypotheses, the analysis provides some intuitive insight justifying their use. Implementation of a posteriori estimates based on hierarchical bases is usually simple, especially in codes that employ hierarchical bases for other purposes, as in the p version of the finite element method. The following discussion is given for the case of continuous piecewise linear approximation, although many of the remarks apply, with appropriate modification, to higher order approximations as well.

In the authors' experience, the cost and accuracy of the hierarchical basis error estimates is comparable to other common a posteriori error estimation schemes in use, for example, those based on the solution of local Neumann [10],[7] or local Dirichlet [4] [17] problems.

Local Neumann error estimates are associated with *elements* and are usually discontinuous at element boundaries. For triangular meshes, these estimates involve solving 3×3 linear systems of equations for the error within each element. In terms global error estimation, under certain conditions, the local Neumann error estimator are asymptotically exact, see Durán and Rodríguez [14], and Durán, Muschietti and Rodríguez [13]. In practice, we have often observed somewhat better $\mathcal{H}^1$ and $\mathcal{L}^2$ norm effectivity ratios [4] when compared to the hierarchical bases error estimates, but the differences are usually not great.

Local Dirichlet error estimates are associated with *nodes* and involve the solution of a small problem on the patch of elements sharing a given vertex in the mesh. Since each triangle generally participates in several such problems, one must analyze the effect of these overlapping subregions [4], [5] in defining a global estimate based on these local computations. Quantitative comparison in terms of effectivity ratios is again unknown, at least to the authors, although we expect the comparison to be similar to that between hierarchical basis estimators and Neumann estimators.

In addition to providing global error estimates, we have used the hierarchical basis estimates to interpolate solution values at newly created vertices in an adaptive mesh refinement algorithm. Rather than solving the large system of equations after every refinement step, interpolated solution values are used to determine subsequent refinements. It is important to assign function values to newly created vertices by a more sophisticated process than simple linear interpolation, since these values will generally influence subsequent refinement steps. Several refinement steps are then taken until the dimension of the finite element space has increased enough to justify the expense of resolving the global problem. Since the hierarchical basis functions are just the quadratic bump functions associated with the *edge* midpoint, the error when added to the linear interpolant of the solution provides a good function value for the vertex. To use the local Neumann and Dirchlet estimates in this way, some average of these errors is needed to define the midpoint solution value.

REFERENCES

[1] D. N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, 21 (1984), pp. 337–344.

[2] A. K. Aziz and I. Babuška, *Part I, survey lectures on the mathematical foundations of the finite element method*, in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Academic Press, New York, 1972, pp. 1–362.

[3] I. Babuška and W. Gui, *Basic principles of feedback and adaptive approaches in the finite element method*, tech. report, Institute for Physical Science and Technology, University of Maryland, College Park, 1985.

[4] I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[5] ———, *A posteriori error estimates for the finite element method*, Internat. J. Numer. Methods Engrg., 12 (1978), pp. 1597–1615.

[6] I. Babuška, O. C. Zienkiewicz, J. Gago, and E. R. de A. Oliveira, *Accuracy Estimates and Adaptive Refinements in Finite Element Computations*, John Wiley and Sons, New York, 1986.

[7] R. E. Bank, *PLTMG: A Software Package for Solving Elliptic Partial Differential Equations, Users' Guide 6.0*, Frontiers Appl. Math., SIAM, Philadelphia, PA, 1990.

[8] R. E. Bank and T. F. Dupont, *Analysis of a two level scheme for solving finite element equations*, Tech. Report CNA-159, Center for Numerical Analysis, University of Texas at Austin, 1980.

[9] R. E. Bank, T. F. Dupont, and H. Yserentant, *The hierarchical basis multigrid method*, Numer. Math., 52 (1988), pp. 427–458.

[10] R. E. Bank and A. Weiser, *Some a posteriori error estimators for partial differential equations*, Math. Comp., 44 (April 1985), pp. 283–301.

[11] M. Benbourenane, *The Hierarchical Basis Multigrid Method for Convection-Dominated Convection-Diffusion Equations*, PhD thesis, University of California at San Diego, 1991.

[12] P. Duflhard, P. Leinen, and H. Yserentant, *Concepts of an adaptive hierarchical finite element code*, IMPACT of Comput. in Sci. and Eng., 1 (1989), pp. 3–35.

[13] R. Durán, M. A. Muschietti, and R. Rodríguez, *On the asymptotic exactness of error estimators for linear triangular finite elements*, Numer. Math., 59 (1991), pp. 107–127.

[14] R. Durán and R. Rodríguez, *On the asymptotic exactness of Bank-Weiser's estimator*, tech. report, Departamento de Matemática, Universidad Nacional de La Plata, Argentina, 1991.

[15] V. Eijkhout and P. Vassilevski, *The role of the strengthened Cauchy-Buniakowskii-Schwarz inequality in multilevel methods*, SIAM Rev., 33 (1991), pp. 405–419.

[16] J. F. Maitre and F. Musy, *The contraction number of a class of two level methods; an exact evaluation for some finite element subspaces and model problems*, in Multigrid Methods: Proceedings, Cologne 1981, Lecture Notes in Mathematics 960, Springer-Verlag, Heidelberg, 1982, pp. 535–544.

[17] C. Mesztenyi and W. Szymczak, *FEARS user's manual for UNIVAC 1100*, Tech. Report BN-991, Institute for Physical Science and Technology, University of Maryland, College Park, 1982.

[18] R. F. Santos, *Moving Space-Time Finite Element Methods for Convection-Diffusion Problems*, PhD thesis, University of California at San Diego, 1991.

[19] A. H. Schatz, *An observation concerning Ritz-Galerkin methods with indefinite bilinear forms*, Math. Comp., 28 (1974), pp. 959–962.

[20] A. Weiser, *Local-Mesh, Local-Order, Adaptive Finite Element Methods with A-Posteriori Error Estimators for Elliptic Partial Differential Equations*, PhD thesis, Yale University, 1981.

[21] O. C. Zienkiewicz and A. Craig, *Adaptive refinement, error estimates, multigrid solution and hierarchical finite element method concepts*, in Accuracy Estimates and Adaptive Refinements in Finite Element Computations, John Wiley and Sons, New York, 1986, pp. 25–59.

[22] O. C. Zienkiewicz, D. W. Kelley, J. de S. R. Gago, and I. Babuška, *Hierarchical finite element approaches, adaptive refinement, and error estimates*, in The Mathematics of Finite Elements and Applications, Academic Press, New York, 1982, pp. 313–346.