The generalized hierarchical basis two-level method for the convection-diffusion equation on a regular grid

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Abstract. We make a theoretical analysis of the application of the generalized hierarchical basis multigrid method to the convection-diffusion equation, discretized using the Scharfetter-Gummel discretization. Our analysis is performed for two levels of grid refinement in which we compare the effects of different interpolation factors for the coarse grid basis functions on the method. In particular, we find the asymptotic convergence rates for the Scharfetter-Gummel- and the ILU-factors. The ILU-factors produce convergence rates independent of the convection directions but dependent on the size of the convection vector. Numerical results illustrating these rates are given.

1 Introduction

Hierarchical basis methods define a robust class of algorithms for solving elliptic partial differential equations, especially for large systems arising in conjunction with adaptive local mesh refinement techniques. They are strongly related to classical multigrid methods, except that only a subset of the unknowns is smoothed during the smoothing steps. As with typical multigrid methods, classical hierarchical basis methods are usually defined in terms of an underlying refinement structure of a sequence of nested meshes.

In recent years, the hierarchical bases have been generalized to completely unstructured meshes, allowing the HBMG and related methods to be successfully applied [9,10,17,18,11,8]. This is done by recognizing the strong connection between the HBMG method and an Incomplete LU factorization of the nodal basis stiffness matrix. Most of the work up to now has been for self adjoint positive definite problems.

In the classical HBMG algorithm, coarse grid basis functions are formed by certain linear combinations of fine grid basis functions. Here the combination coefficients are derived from the geometry of the mesh.

⋆ The work of this author was supported by the U. S. Office of Naval Research under contract N00014-89J-1440.

⋆⋆ The work of this author was supported by a DAAD-fellowship HSPII from the German Federal Ministry for Education, Science, Research and Technology.

In this paper we are concerned with the construction of generalized hierarchical basis functions using expansion coefficients derived in a more algebraic fashion. Different choices of expansion coefficients typically have no effect on the supports of the basis functions but do have a profound effect on the shape of the basis functions themselves, and hence on the numerical values appearing in the stiffness matrix with respect to the hierarchical basis. One can even choose different coefficients for test and trial spaces, similar to Petrov-Galerkin approximations. The HBMG iteration itself is just a block Gauß-Seidel iteration applied to the stiffness matrix in the generalized hierarchical basis representation.

We remark that computationally it is undesirable to assemble and solve the set of equations given in the hierarchical basis since the matrix is less sparse than the corresponding matrix in nodal basis representation. In practice, hierarchical basis methods are implemented using the standard nodal basis, in combination with some recursive algorithms that are very similar to the standard multigrid V-Cycle [5,16]. Here we are mainly concerned with obtaining estimates for the rate of convergence for hierarchical basis methods. The presentation is mainly restricted to the two level case. While most of our results apply to general triangulations of shape regular elements, our analysis in section 5 is restricted to the case of a uniform mesh of isosceles right triangles, where the Scharfetter-Gummel discretization can be interpreted as a five-point difference operator.
We will discuss several alternatives for choosing the interpolation coefficients for the fine grid nodes: The 'trivial' Gauß-Seidel factors given by 0, the Scharfetter-Gummel and the ILU-interpolation factors. We also suggest a hybrid variant that combines the best properties of the Scharfetter-Gummel and ILU coefficients.

Early work on multigrid methods for convection diffusion problems includes [15,20,12]. The application of classical HBMG to such problems was considered in [21,2]. A study of two point boundary values problems which motivated the present work is given in [6].

The rest of the paper is organized as follows: In section 2, we give a brief description of the model problem, finite element spaces and the construction of the generalized hierarchical basis functions. In section 3 we will derive the Scharfetter-Gummel and the ILU interpolation coefficients. Some auxiliary results are stated in section 4. In section 5 we will analyse the element stiffness matrix represented in the hierarchical basis on a reference triangle. We will introduce the hierarchical basis two-level method in section 6, and we will use the results of the previous sections to derive asymptotic convergence rates for the resulting methods. Numerical experiments and some comments on the generalization of the method to the multilevel case are reported in section 7.

2 The model problem

We consider the two dimensional convection-diffusion equation

$$-\nabla(\nabla u + \beta u) = f \quad \text{in } \Omega \tag{1}$$

with boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

where Ω is a polygonal domain and β is constant. Let $H^1(\Omega)$ be the standard Sobolev space consisting of square integrable functions with square integrable derivatives of first order and let $H_0^1(\Omega)$ be a subspace of $H^1(\Omega)$ consisting of functions that vanish on $\partial\Omega$. It is well known that $u \in H_0^1(\Omega)$ is the solution of

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega)$$

where $f(v) = \int_\Omega fv\ dx$ and $a(u, v) = \int_\Omega \nabla v(\nabla u + \beta u)\ dx$.

We discretize (1),(2) with the Scharfetter-Gummel quasi-uniform triangulation $\mathcal{T}$ of Ω, consisting of shape-regular triangles characterized by a small parameter h. For simplicity we assume that the triangulation has no obtuse angles, although this is mainly for convenience (see [7]). Corresponding to the triangulation $\mathcal{T}$, let $\mathcal{V} \subset H_0^1(\Omega)$ be the finite element trial space consisting of continuous piecewise linear functions vanishing on the boundary. For each vertex v_i, we can associate a box b_i, generated by the perpendicular bisectors of the triangle edges incident on that vertex. Let $\mathcal{W}$ be the test space of (discontinuous) piecewise constant functions with respect to the boxmesh, vanishing inside those boxes that are associated to boundary vertices. We now integrate equation (1) over the box b_i, and then apply the divergence theorem to get

$$-\int_{\partial b_i} (\nabla u + \beta u) \cdot n\ ds = 0$$

where n is the (edgewise) outward normal for the box b_i. Let ϕ be defined such that $\beta = \nabla\phi$ holds. Along the box boundary segment between vertices v_1 and v_2, the flux term

$$-(\nabla u + \beta u) \cdot n = -e^{-\phi}\nabla(e^{\phi}u) \cdot n$$

is approximated by

$$e^{-\tilde{\phi}_3}\left(\frac{e^{\phi_1}u_1 - e^{\phi_2}u_2}{l_3}\right)$$

where $\phi_i = \phi(v_i)$, $u_i = u(v_i)$ and $l_3 = |v_1 - v_2|$. The value of $\tilde{\phi}_3$ is given by

$$e^{-\tilde{\phi}_3} = \frac{\phi_1 - \phi_2}{e^{\phi_1} - e^{\phi_2}}.$$

Since ϕ is linear, we have $\phi_1 - \phi_2 = \langle \beta, v_1 - v_2 \rangle$. These equations define the Scharfetter-Gummel discretization of (1). Following [3], we can form an *element stiffness matrix* for the Scharfetter-Gummel discretization, which is given by

$$
\begin{aligned}
A &= A_s D \\
&= \begin{pmatrix} \mathcal{B}_{21}L_3 + \mathcal{B}_{31}L_2 & -\mathcal{B}_{12}L_3 & -\mathcal{B}_{13}L_2 \\ -\mathcal{B}_{21}L_3 & \mathcal{B}_{12}L_3 + \mathcal{B}_{32}L_1 & -\mathcal{B}_{23}L_1 \\ -\mathcal{B}_{31}L_2 & -\mathcal{B}_{32}L_1 & \mathcal{B}_{13}L_2 + \mathcal{B}_{23}L_1 \end{pmatrix}
\end{aligned} \tag{3}
$$

where

$$
A_s = \begin{pmatrix} e^{-\tilde{\phi}_3}L_3 + e^{-\tilde{\phi}_2}L_2 & -e^{-\tilde{\phi}_3}L_3 & -e^{-\tilde{\phi}_2}L_2 \\ -e^{-\tilde{\phi}_3}L_3 & e^{-\tilde{\phi}_3}L_3 + e^{-\tilde{\phi}_1}L_1 & -e^{-\tilde{\phi}_1}L_1 \\ -e^{-\tilde{\phi}_2}L_2 & -e^{-\tilde{\phi}_1}L_1 & e^{-\tilde{\phi}_1}L_1 + e^{-\tilde{\phi}_2}L_2 \end{pmatrix},
$$

$$
D = \begin{pmatrix} e^{\phi_1} & 0 & 0 \\ 0 & e^{\phi_2} & 0 \\ 0 & 0 & e^{\phi_3} \end{pmatrix}.
$$

Here $\mathcal{B}_{ij} \equiv \mathcal{B}(\phi_i - \phi_j) = \mathcal{B}(\langle \beta, v_i - v_j \rangle)$, where $\mathcal{B}(x) = x/(e^x - 1)$ is the *Bernoulli function*, and L_i are the matrix elements arising in the corresponding element stiffness matrix for the Poisson equation ($\beta = 0$) for the case of piecewise linear finite elements using the standard nodal basis. A more detailed description is given in [3]. We denote the (global) bilinear form for the Scharfetter-Gummel discretization by $a^{SG}(\cdot, \cdot)$ which is defined on $\mathcal{V} \times \mathcal{W}$.

The spaces $\mathcal{V}$ and $\mathcal{W}$ have natural nodal bases $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ that satisfy

$$
\begin{aligned}
\phi_i(v_j) &= \delta_{ij} \quad \forall i, j = 1, \cdots, n, \\
\psi_i(v_j) &= \delta_{ij} \quad \forall i, j = 1, \cdots, n,
\end{aligned}
$$

where $\{v_i, i = 1, \cdots n\}$ is the set of all interior vertices of the triangulation $\mathcal{T}$. With these basis functions, the finite element solution can be written as $u = \sum x_i \phi_i$ where $x = (x_i)_{i=1,\cdots,n}$ satisfies

$$
A^{NB} x = b.
$$

$A_{ij}^{NB} = a^{SG}(\phi_j, \psi_i)$ is the stiffness matrix with respect to the nodal basis, and b is given by an appropriate linear functional f: $b_i = f(\psi_i)$.

The first step in constructing a hierarchical basis is to create a certain hierarchical structure based on the given triangulation $\mathcal{T}$. We will consider the case of two nested meshes where the fine mesh is a uniform refinement of a coarse mesh, generated by pairwise connecting the midpoints of the coarse grid edges in the usual way [5], [22], [16]. Here we can make the direct sum decomposition $X = X_c \oplus X_f$, where X_c is the set of (interior) coarse grid vertices, and X_f is the set of (interior) fine grid vertices (those not in X_c). For each vertex $v_i \in X_f$, there is a unique pair of vertex parents $v_j, v_k \in X_c$ such that v_i is the midpoint of the edge connecting v_j and v_k ($v_i = (v_j + v_k)/2$).

Consistent with the decomposition of X, we define the hierarchical decomposition

$$\mathcal{V} = V_c \oplus V_f$$

where

$$V_f = \{\phi \in \mathcal{V} | \phi(x) = 0 \text{ for all coarse grid vertices } x\}$$

and V_c is a space spanned by basis functions associated with the coarse grid vertices. In the case of the standard nodal basis, V_c is given by

$$V_c = \{\phi \in \mathcal{V} | \phi(x) = 0 \text{ for all fine grid vertices } x\}.$$

For the classical hierarchical basis, V_c is the space of continuous piecewise linear functions on the coarse grid. In the generalized hierarchical basis method we modify V_c in order to improve the convergence behaviour of the HBMG method. In all cases, the basis functions of V_c are linear combinations of the (fine grid) nodal basis functions. In the nodal basis, the combination coefficients associated with fine grid vertices are simply 0, i.e. the coarse grid basis functions are chosen equal to the fine grid basis functions.

In the classical hierarchical basis, the coefficients are derived from the geometry of the mesh. In section 3 we will derive the Scharfetter-Gummel and the ILU coefficients. The Scharfetter-Gummel coefficients depend not only on the geometry of the mesh, but also on the boundary value problem itself, in particular on the convection vector β.

A more algebraic approach leads to the ILU coefficients. Here the strong connection between the HBMG method and the ILU factorization is explored, and the coefficients are chosen to eliminate certain off-diagonal elements of the hierarchical basis stiffness matrix, as is done in the case of classical ILU elimination.

The hierarchical basis functions for the test space $\mathcal{W}$ are defined in a similar fashion.

3 Derivation of the Scharfetter-Gummel and the ILU coefficients

3.1 The Scharfetter-Gummel coefficients

From the Scharfetter-Gummel formula, an exponential interpolation scheme can be derived. We begin by noting that fundamental solutions of our model convection-diffusion equation (1) are given by

$$u(x) = \alpha + \gamma e^{-\langle \beta, x \rangle}$$

for some constants $\alpha, \gamma \in \mathbb{R}$. Suppose the values $u_1 = u(v_1)$ and $u_2 = u(v_2)$ are known and $u_m \equiv u(v_m) = u(\theta v_1 + (1-\theta)v_2)$ is to be approximated for a vertex v_m between v_1 and v_2. If we require exact interpolation of the fundamental solutions on the one dimensional edge between v_1 and v_2, we obtain by a straightforward calculation $u_m = \nu u_1 + \tilde{\nu} u_2$ where

$$\nu = \frac{\theta \mathcal{B}(\langle \beta, v_2 - v_1 \rangle)}{\mathcal{B}(\theta \langle \beta, v_2 - v_1 \rangle)}, \tag{4}$$

$$\tilde{\nu} = 1 - \nu. \tag{5}$$

For $\beta = 0$ this method reduces to the classical HBMG algorithm ($\nu = \tilde{\nu} = \frac{1}{2}$).

For $\beta \neq 0$ an equivalent formulation for ν is given by

$$\nu = \frac{e^{\theta \langle \beta, v_2 - v_1 \rangle} - 1}{e^{\langle \beta, v_2 - v_1 \rangle} - 1} = \frac{\mathcal{B}(\theta \langle \beta, v_2 - v_1 \rangle)}{\mathcal{B}(\theta \langle \beta, v_2 - v_1 \rangle) + \mathcal{B}(\theta \langle \beta, v_1 - v_2 \rangle)}.$$

In the case of regular refinement we have $\theta = 1/2$. Note that the interpolation coefficients lie in $(0, 1)$ and sum up to 1: $\nu + \tilde{\nu} = 1$.

3.2 The ILU coefficients

Unless we choose all coefficients to be equal to zero (which would lead to the classical nodal basis block Gauß-Seidel iteration), the supports of the coarse grid hierarchical basis functions are larger than those of the nodal basis functions. This leads to a less sparse stiffness matrix. By analysing the computation of the numerical values in the hierarchical basis stiffness matrix, we can derive coefficients that force certain values to be zero, in particular those corresponding to vertical or horizontal edges between fine and coarse grid vertices in the two level grid.

To derive the coefficients, let v_a be a coarse grid vertex and let v_i, $i = 1, \cdots, 6$ be the (fine grid) neighbours of v_a as shown in Figure 1.

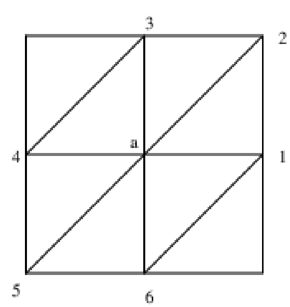

Fig. 1. Triangulation with coarse grid vertex v_a and fine grid vertices v_i, $i = 1, \cdots, 6$

The hierarchical basis function corresponding to v_a is a linear combination of the (fine grid) nodal basis function corresponding to v_a and the (fine grid) nodal basis functions corresponding to the neighbouring fine grid vertices

$v_i,\ i = 1, \cdots, 6$. Let ϕ_a and $\phi_i,\ i = 1, \cdots, 6$ denote the relevant nodal basis functions. The hierarchical basis function $\hat{\phi}_a$ corresponding to v_a is then given by the linear combination

$$\hat{\phi}_a = \phi_a + \sum_{i=1}^{6} \theta_i^a \phi_i$$

with some coefficients $\theta_i^a,\ i = 1, \cdots, 6$. For example, the value in the hierarchical basis stiffness matrix corresponding to the edge between v_a and v_1 will be

$$\begin{aligned} a^{SG}(\hat{\phi}_a, \phi_1) &= a^{SG}(\phi_a + \sum_{i=1}^{6} \theta_i^a \phi_i, \phi_1) \\ &= a^{SG}(\phi_a, \phi_1) + \theta_1^a a^{SG}(\phi_1, \phi_1) + \theta_2^a a^{SG}(\phi_2, \phi_1). \end{aligned}$$

Here we used the fact that the Scharfetter-Gummel discretization leads to a five point stencil. We can force $a^{SG}(\hat{\phi}_a, \phi_1)$ to be zero by choosing

$$\theta_1^a = -\frac{a^{SG}(\phi_a, \phi_1)}{a^{SG}(\phi_1, \phi_1)}, \tag{6}$$

$$\theta_2^a = -\frac{a^{SG}(\phi_a, \phi_2)}{a^{SG}(\phi_1, \phi_1)} = 0. \tag{7}$$

Analogously, the remaining coefficients can be determined. We refer to these factors as the ILU coefficients since they lead to the elimination of certain off-diagonal elements of the hierarchical basis stiffness matrix. This scheme leads to a local minimization of the affected row and column vectors of the stiffness matrix at each elimination step. Note that the ILU coefficients can have either sign, and generally $\theta_m^a + \theta_m^b \neq 1$ where the fine grid vertex v_m is the midpoint of the coarse grid vertices v_a and v_b.

4 Some auxiliary results

Our analysis of the two level method will be framed in terms of a strengthened Cauchy-Schwarz inequality, a fairly traditional approach [19,13,4]. In this section, we collect some technical results which are necessary for this analysis in subsequent sections. We begin with a simple lemma from linear algebra:

Lemma 1. *(angle between order 1 subspaces) Let $v \in \mathbb{R}^n, w \in \mathbb{R}^m$ and $C = vw^T \in \mathbb{R}^{n\times m}$. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be symmetric, positive semi-definite with $v \in range(A)$ and $w \in range(B)$. Then there exists a positive constant γ such that for every $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^m$*

$$|x^T C y| \leq \gamma \sqrt{x^T A x} \sqrt{y^T B y}$$

where

$$\gamma = \sqrt{v^T A^+ v}\sqrt{w^T B^+ w}$$

and A^+ *is the (generalized) inverse of* A *restricted to* $range(A)$.

Proof. see [2] Lemma 2.1

Lemma 2. *(angle between subspaces of any order) Let* A *be* $n \times n$, *symmetric positive semi-definite. Let* B *be* $m \times m$, *symmetric positive semi-definite. Let* C *be* $n \times m$ *and satisfy* $Kernel(A) \subseteq Kernel(C^t)$ *and* $Kernel(B) \subseteq Kernel(C)$. *Let* γ *be*

$$\gamma = \max_{\substack{x^t A x = 1 \\ y^t B y = 1}} |x^T C y|.$$

Define matrices $D = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$, $R = \begin{pmatrix} I \\ 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times n}$, $S = \begin{pmatrix} 0 \\ I \end{pmatrix} \in \mathbb{R}^{(n+m)\times m}$. *This leads to*

$$x^T C y = x^T R^T D S y \quad \text{and} \quad x^T A x = x^T R^T D R x,\ y^T B y = y^T S^T D S y.$$

Then γ *is the angle between the subspace pair* $\{range(D^{\frac{1}{2}}R), range(D^{\frac{1}{2}}S)\}$ *and thus the largest singular value of* $C := Q^T_{D^{\frac{1}{2}}R} Q_{D^{\frac{1}{2}}S}$ *where* $Q_{D^{\frac{1}{2}}R}$ *and* $Q_{D^{\frac{1}{2}}S}$ *are given by the QR-decompositions of* $D^{\frac{1}{2}}R$ *and* $D^{\frac{1}{2}}S$:

$$D^{\frac{1}{2}}R = Q_{D^{\frac{1}{2}}R} R_{D^{\frac{1}{2}}R}, \qquad Q^T_{D^{\frac{1}{2}}R} Q_{D^{\frac{1}{2}}R} = I,$$
$$D^{\frac{1}{2}}S = Q_{D^{\frac{1}{2}}S} R_{D^{\frac{1}{2}}S}, \qquad Q^T_{D^{\frac{1}{2}}S} Q_{D^{\frac{1}{2}}S} = I,$$

Proof. see [14] p. 429

Lemma 3. *Let* $\mathcal{V}$ *be the space of continuous piecewise linear polynomials associated with the triangulation* $\mathcal{T}$. *Let* $\mathcal{V} = V_c \oplus V_f$ *be a decomposition of* $\mathcal{V}$. *Let*

$$b(v,w) = \sum_{t \in \mathcal{T}} b(v,w)_t$$

be an inner product defined in $\mathcal{V}$, *with induced norm*

$$\|u\|^2 = \sum_{t \in \mathcal{T}} b(u,u)_t = \sum_{t \in \mathcal{T}} \|u\|_t^2$$

Suppose for each $t \in \mathcal{T}$ *exists* $0 \le \gamma_t < 1$ *such that for all* $v \in V_c$ *and for all* $w \in V_f$

$$|b(v,w)_t| \le \gamma_t\ \|v\|_t\ \|w\|_t.$$

Then

$$|b(v,w)| \le \gamma\ \|v\|\ \|w\| \tag{8}$$

with

$$\gamma = max_{t \in \mathcal{T}} \gamma_t.$$

Proof. see [2] Lemma 2.3

Lemma 4. *Let $A(\beta)$ be the nodal basis stiffness matrix for our model problem (1) discretized by the Scharfetter-Gummel method. Then there exists a non-singular diagonal matrix B such that AB is symmetric.*

Proof. Let

$$B = diag(e^{-\langle \beta, v_i \rangle}), \tag{9}$$

The Lemma immediately follows from the form of the element stiffness matrices given in (3). □

This leads to the following corollary:

Corollary 5. *Let A be the nodal basis stiffness matrix for our model problem (1) discretized by the Scharfetter-Gummel method, and let B be defined in (9). Then $B^{-\frac{1}{2}}AB^{\frac{1}{2}}$ is a symmetric matrix.*

For the remaining lemmas in this section, we restrict our attention to the special case of a uniform mesh composed of isosceles right triangles. For this case, the Scharfetter-Gummel discretization can be interpreted as a 5-point difference approximation.

Lemma 6. *Let $A(\beta)$ be the nodal basis stiffness matrix for our model problem (1) discretized by the Scharfetter-Gummel method. Then*

$$A(\beta) = A(-\beta)^T.$$

Proof. $A(\beta)$ can be described by the 5 point stencil

$$\begin{pmatrix} & \mathcal{B}(-\beta_2 h) & \\ \mathcal{B}(\beta_1 h) & d & \mathcal{B}(-\beta_1 h) \\ & \mathcal{B}(\beta_2 h) & \end{pmatrix} \qquad \text{with} \tag{10}$$

$$d = \mathcal{B}(\beta_1 h) + \mathcal{B}(-\beta_1 h) + \mathcal{B}(\beta_2 h) + \mathcal{B}(-\beta_2 h) \tag{11}$$

where $\mathcal{B}(x)$ denotes the Bernoulli function. It follows that $A(\beta) = A(-\beta)^T$. □

Corollary 7. *Let $A(\beta)$ be the nodal basis stiffness matrix for our model problem (1) discretized by the Scharfetter-Gummel method. Let B be the non-singular diagonal matrix defined in (10). Then*

$$B^{-\frac{1}{2}}A(\beta)B^{\frac{1}{2}} = B^{\frac{1}{2}}A(-\beta)B^{-\frac{1}{2}}. \tag{12}$$

Remark 8. Another useful notation of (12) is

$$B^{-\frac{1}{2}}A(\beta)B^{\frac{1}{2}} = \frac{1}{2}[B^{-\frac{1}{2}}A(\beta)B^{\frac{1}{2}} + B^{\frac{1}{2}}A(-\beta)B^{-\frac{1}{2}}]. \tag{13}$$

This remark will allow us to analyse the element stiffness matrices for the term on the *rhs* of (13). We will use Lemma 1 to get the factors γ_t in the strengthened Cauchy inequality (8) for the reference triangle, and then extend the result to the domain Ω using Lemma 3.

5 The element stiffness matrix

In this section, we make estimates for γ_t for our interpolation schemes for the special case of a uniform mesh. We consider a coarse grid reference triangle $\hat{t}$ with the coarse grid vertices $v_1 = (0,0)$, $v_2 = (2,0)$, $v_3 = (0,2)$ and the fine grid vertices $v_a = (1,1)$, $v_b = (0,1)$, $v_c = (1,0)$ as illustrated in Fig. 1.

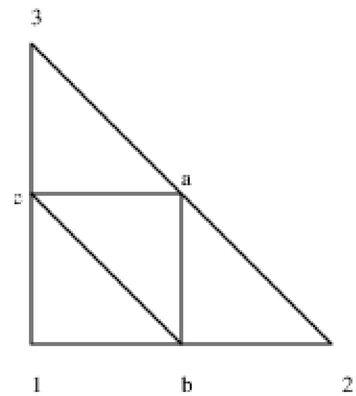

Fig. 2. A coarse grid triangle.

The restriction of the space $\mathcal{V}$ to $\hat{t}$ is the space of piecewise linear functions on $\hat{t}$, and the nodal basis functions $\phi_1, \cdots, \phi_c$ are those associated with the vertices $v_1, \cdots, v_c$. $a_t^{SG}(\cdot,\cdot)$ denotes the Scharfetter-Gummel bilinear form evaluated on the reference triangle.
The entries of the stiffness matrix A^{NB} can be regarded as the sum of the contributions of the coarse grid triangles:

$$A_{ij}^{NB} = \sum_{t \in \Omega} a_t^{SG}(\phi_j, \phi_i).$$

The nodal basis element stiffness matrix for the Scharfetter Gummel discretization on the reference triangle is given by

$$A_{\hat{t}}^{NB} = \begin{pmatrix} A_f^{NB} & A_{fc}^{NB} \\ A_{cf}^{NB} & A_c^{NB} \end{pmatrix} \qquad \text{with}$$

$$A_f^{NB} = \begin{pmatrix} a_t(\phi_a,\phi_a) & a_t(\phi_a,\phi_b) & a_t(\phi_a,\phi_c) \\ a_t(\phi_b,\phi_a) & a_t(\phi_b,\phi_b) & 0 \\ a_t(\phi_c,\phi_a) & 0 & a_t(\phi_c,\phi_c) \end{pmatrix},$$

$$A_{fc}^{NB} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{B}(-\beta_2 h) & 0 & -\frac{1}{2}\mathcal{B}(\beta_2 h) \\ -\frac{1}{2}\mathcal{B}(-\beta_1 h) & -\frac{1}{2}\mathcal{B}(\beta_1 h) & 0 \end{pmatrix},$$

$$A_{cf}^{NB} = \begin{pmatrix} 0 & -\frac{1}{2}\mathcal{B}(\beta_2 h) & -\frac{1}{2}\mathcal{B}(\beta_1 h) \\ 0 & 0 & -\frac{1}{2}\mathcal{B}(-\beta_1 h) \\ 0 & -\frac{1}{2}\mathcal{B}(-\beta_2 h) & 0 \end{pmatrix},$$

$$A_{c}^{NB} = \begin{pmatrix} \frac{1}{2}\mathcal{B}(\beta_1 h) + \frac{1}{2}\mathcal{B}(\beta_2 h) & 0 & 0 \\ 0 & \frac{1}{2}\mathcal{B}(-\beta_1 h) & 0 \\ 0 & 0 & \frac{1}{2}\mathcal{B}(-\beta_2 h) \end{pmatrix}$$

where

$$\begin{aligned} a_t(\phi_a, \phi_a) &= \mathcal{B}(-\beta_1 h) + \mathcal{B}(-\beta_2 h), \\ a_t(\phi_a, \phi_b) &= -\mathcal{B}(-\beta_1 h), \\ a_t(\phi_a, \phi_c) &= -\mathcal{B}(-\beta_2 h), \\ a_t(\phi_b, \phi_a) &= -\mathcal{B}(\beta_1 h), \\ a_t(\phi_b, \phi_b) &= \mathcal{B}(\beta_1 h) + \frac{1}{2}\mathcal{B}(-\beta_2 h) + \frac{1}{2}\mathcal{B}(\beta_2 h), \\ a_t(\phi_c, \phi_a) &= -\mathcal{B}(\beta_2 h), \\ a_t(\phi_c, \phi_c) &= \mathcal{B}(\beta_2 h) + \frac{1}{2}\mathcal{B}(-\beta_1 h) + \frac{1}{2}\mathcal{B}(\beta_1 h). \end{aligned}$$

Here $\mathcal{B}(x)$ denotes the Bernoulli function. A_t^{NB} is positive (semi)-definite (to see this use Gerschgorin criterion). Let

$$B = diag(e^{(\beta_1+\beta_2)h}, e^{\beta_2 h}, e^{\beta_1 h}, 1, e^{2\beta_1 h}, e^{2\beta_2 h}).$$

Then by Lemma 6

$$A_{t,symm}^{NB} := B^{-\frac{1}{2}} A_t^{NB} B^{\frac{1}{2}}$$

is a symmetric (semi-) positive definite matrix with blocks of the form

$$\begin{aligned} A_{fc,symm}^{NB} &= (A_{cf,symm}^{NB})^T \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2}e^{\frac{\beta_2 h}{2}}\mathcal{B}(\beta_2 h) & 0 & -\frac{1}{2}e^{\frac{\beta_2 h}{2}}\mathcal{B}(\beta_2 h) \\ -\frac{1}{2}e^{\frac{\beta_1 h}{2}}\mathcal{B}(\beta_1 h) & -\frac{1}{2}e^{\frac{\beta_1 h}{2}}\mathcal{B}(\beta_1 h) & 0 \end{pmatrix} \\ A_{c,symm}^{NB} &= A_c^{NB} \\ A_{f,symm}^{NB} &= \begin{pmatrix} a_t(\phi_a, \phi_a) & -e^{\frac{\beta_1 h}{2}}\mathcal{B}(\beta_1 h) & -e^{\frac{\beta_2 h}{2}}\mathcal{B}(\beta_2 h) \\ -e^{\frac{\beta_1 h}{2}}\mathcal{B}(\beta_1 h) & a_t(\phi_b, \phi_b) & 0 \\ -e^{\frac{\beta_2 h}{2}}\mathcal{B}(\beta_2 h) & 0 & a_t(\phi_c, \phi_c) \end{pmatrix}. \end{aligned}$$

The crucial point in the following analysis is the observation that although the symmetrized element stiffness matrices for β and $-\beta$ are not the same, their sum over all triangles is the same, i.e. $A(\beta)_{symm}^{NB} = A(-\beta)_{symm}^{NB}$ (see

Corollary 7). From now on we can thus consider the averaged element stiffness matrices

$$A^{NB}_{t,av,symm} = \frac{1}{2}[A(\beta)^{NB}_{t,symm} + A(-\beta)^{NB}_{t,symm}]$$

for which we have

$$A^{NB}_{symm} = \sum_{t\in\Omega} A^{NB}_{t,av,symm}.$$

Note that $A^{NB}_{t,f,av,symm}$ and $A^{NB}_{t,c,av,symm}$ are non-singular.
To transform these averaged element stiffness matrices from nodal basis to hierarchical basis representation, we choose transformation matrices S of the form

$$S = \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \qquad \text{where} \quad R = \begin{pmatrix} 0 & \nu & \nu \\ \eta & 0 & \eta \\ \theta & \theta & 0 \end{pmatrix}.$$

The values of θ, η and ν for the three schemes we consider are given in Table 1. Note that these values are obtained by taking the coefficients derived in section 3 and applying the diagonal similarity transformation used in computing $A^{NB}_{t,symm}$.

coefficient	G-S	S-G	ILU
θ	0	$\frac{e^{\frac{\beta_1 h}{2}}\mathcal{B}(\beta_1 h)}{\mathcal{B}(\beta_1 h)+\mathcal{B}(-\beta_1 h)}$	$\frac{e^{\frac{\beta_1 h}{2}}\mathcal{B}(\beta_1 h)}{\mathcal{B}(\beta_1 h)+\mathcal{B}(-\beta_1 h)+\mathcal{B}(\beta_2 h)+\mathcal{B}(-\beta_2 h)}$
η	0	$\frac{e^{\frac{\beta_2 h}{2}}\mathcal{B}(\beta_2 h)}{\mathcal{B}(\beta_2 h)+\mathcal{B}(-\beta_2 h)}$	$\frac{e^{\frac{\beta_2 h}{2}}\mathcal{B}(\beta_2 h)}{\mathcal{B}(\beta_1 h)+\mathcal{B}(-\beta_1 h)+\mathcal{B}(\beta_2 h)+\mathcal{B}(-\beta_2 h)}$
ν	0	$\frac{e^{\frac{(\beta_2-\beta_1)h}{2}}\mathcal{B}((\beta_2-\beta_1)h)}{\mathcal{B}((\beta_2-\beta_1)h)+\mathcal{B}((\beta_1-\beta_2)h)}$	0

Table 1. Interpolation coefficients for various schemes. "G-S" is Gauß-Seidel, "S-G" is exponential interpolation and "ILU" is ILU interpolation.

We next calculate the averaged element stiffness matrix for the symmetrized problem in hierarchical basis representation:

$$\begin{aligned}
A^{HB}_{t,av,symm} &= S^T A^{NB}_{t,av,symm} S = \begin{pmatrix} A^{HB}_{f,av,symm} & A^{HB}_{fc,av,symm} \\ A^{HB}_{cf,av,symm} & A^{HB}_{c,av,symm} \end{pmatrix}, \\
A^{HB}_{fc,av,symm} &= A^{NB}_{fc,symm} + A^{NB}_{f,symm} R = (A^{HB}_{cf,av,symm})^T, \\
A^{HB}_{f,av,symm} &= A^{NB}_{f,symm}, \\
A^{HB}_{c,av,symm} &= R^T A^{NB}_{f,symm} R + R^T A^{NB}_{fc,symm} + A^{NB}_{cf,symm} R + A^{NB}_{c,symm}.
\end{aligned}$$

Our goal is to compute γ_t in the strengthened Cauchy-Schwarz inequality

$$|x^T A^{HB}_{fc,av,symm} y| \le \gamma_t (x^T A^{HB}_{f,av,symm} x)^{1/2} (y^T A^{HB}_{c,av,symm} y)^{1/2}.$$

If $A^{HB}_{fc,av,symm}$ is a rank one matrix, we define v and w such that

$$A^{HB}_{fc,av,symm} = vw^T.$$

and apply Lemma 1 to determine γ_t as

$$\gamma_t = \sqrt{v^T (A^{HB}_{f,av,symm})^{-1} v} \sqrt{w^T (A^{HB}_{c,av,symm})^{-1} w}. \tag{14}$$

If $A^{HB}_{fc,av,symm}$ is not a rank one matrix, we use Lemma 2 to determine γ_t.

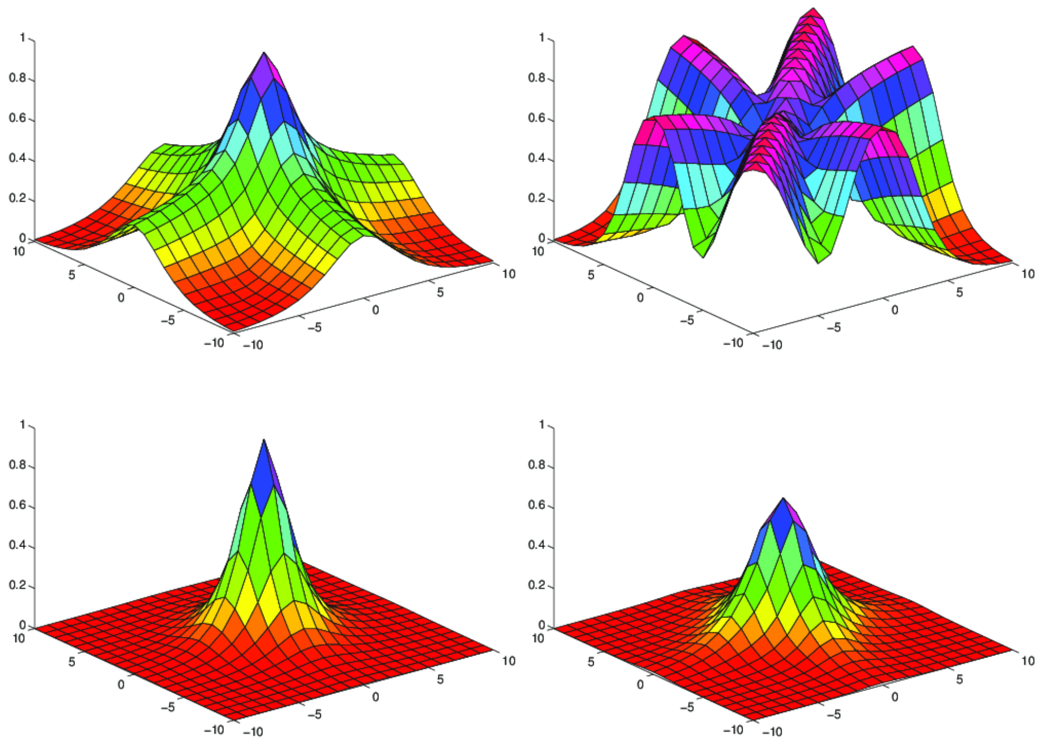

Fig. 3. Top: γ_t for Gauß-Seidel factors (left) and Scharfetter-Gummel factors (right). Bottom: γ_t for ILU factors (left) and the hybrid scheme (right).

Using the interpolation coefficients given in Table 1, we have numerically computed γ_t for the Gauß-Seidel, the ILU and the Scharfetter-Gummel factors. $A^{HB}_{fc,av,symm}$ is a rank one matrix only for the ILU factors.

The results are shown in Figure 3. In these figures, γ_t is graphed as a function of $(\beta_1 h, \beta_2 h)$.

In viewing these figures, we note that the ILU scheme seems preferable in most cases; when $|\beta|h \approx 0$, the convergence rate for ILU approaches one, while that for exponential interpolation approaches $1/\sqrt{2}$. Thus we are motivated to suggest a hybrid variant which combines the best properties of these two methods. Let θ_{SG} and θ_{ILU} be a pair of corresponding interpolation factors for the SG and ILU schemes. Let $F(x)$ be any continuous, monotonically increasing function on $[0,\infty)$ satisfying $F(0) = 0$, i.e. $F(x) = x^p$ for some $p > 0$ or $F(x) = e^{\alpha x} - 1$ for some $\alpha > 0$. Define a new interpolation factor θ by

$$\theta = \frac{F(|\beta|\,|v_2 - v_1|)\theta_{ILU} + \theta_{SG}}{F(|\beta|\,|v_2 - v_1|) + 1}$$

where v_1, v_2 denote the coarse grid vertices between which we want to interpolate. We have a similar formula for all corresponding pairs of interpolation factors.

Near $|\beta|\,|v_2 - v_1| = 0$, this scheme behaves like the exponential scheme, which is the best scheme for small $|\beta|\,|v_2 - v_1|$. On the other hand, when $|\beta|\,|v_2 - v_1|$ is large, it will behave like the ILU scheme. Figure 3 (bottom right) shows γ_t for $F(x) = e^{\frac{x}{2}} - 1$.

6 Analysis for the two-level method

In this section we will derive upper bounds for the convergence rate of the generalized hierarchical basis multigrid method (on two levels).

The discretization described in section 2 leads to a linear system of equations of the form

$$A^{NB}x = b. \tag{15}$$

If we order the nodal basis functions by level (i.e. first coarse and then fine), the following block partitioning results for A^{NB}:

$$A^{NB} = \begin{pmatrix} A_c^{NB} & A_{cf}^{NB} \\ A_{fc}^{NB} & A_f^{NB} \end{pmatrix}, \tag{16}$$

where A_f^{NB} corresponds to the nodal basis functions of the fine grid nodes, A_c^{NB} corresponds to the (fine grid) nodal basis functions of the coarse grid nodes and A_{cf}^{NB} and A_{fc}^{NB} correspond to the coupling between the two sets of basis functions. We consider transformations of the form $A^{HB} = S^T A^{NB} \tilde{S}$, where S and $\tilde{S}$ have the block structure

$$S = \begin{pmatrix} I & 0 \\ R & I \end{pmatrix} \quad \text{and} \quad \tilde{S} = \begin{pmatrix} I & 0 \\ \tilde{R} & I \end{pmatrix}. \tag{17}$$

By direct calculation, we obtain

$$S^T A^{NB} \tilde{S} = \begin{pmatrix} A_c^{HB} & A_{cf}^{NB} + R^T A_f^{NB} \\ A_{fc}^{NB} + A_f^{NB}\tilde{R} & A_f^{NB} \end{pmatrix} \tag{18}$$

where

$$A_c^{HB} = A_c^{NB} + R^T A_{fc}^{NB} + A_{cf}^{NB} \tilde{R} + R^T A_f^{NB} \tilde{R}. \tag{19}$$

Different algorithms are characterized by different choices of R and $\tilde{R}$. Our two level iteration is the block symmetric Gauß-Seidel iteration. Given a starting vector x_0, the matrix formulation for the remaining iterates is given by

$$x_{i+1} = x_i + W^{-1}(b - Ax_i) \tag{20}$$

where $W = (D + U)D^{-1}(D + L)$, and D, L, and U are the block diagonal, lower triangular, and upper triangular, resp. parts of A. The asymptotic convergence rate of the method is given by the spectral radius $\sigma(M)$ of the iteration matrix $M = I - W^{-1}A$.

Lemma 9. *Let A be a regular matrix, let B be a diagonal non-singular matrix and let $\bar{A} = B^{-1}AB$ be a similarity transformation. Let $M, \bar{M}$ be the iteration matrices for the block symmetric Gauß-Seidel iteration (20) applied to $A, \bar{A}$. Then*

$$\sigma(M) = \sigma(\bar{M}),$$

i.e. the asymptotic convergence rates for A and $\bar{A}$ are asymptotically the same.

Proof. The proof is straightforward.

Lemma 9 justifies deriving convergence rates for the symmetrized nodal and hierarchical basis stiffness matrices since the unsymmetric matrices are symmetrized by a similarity transformation with a non-singular diagonal matrix B.

Theorem 10. *Let $\bar{W} = B^{-1}WB$ where W and B are defined as above, and suppose that $\gamma < 1$ in the strengthened Cauchy inequality. Then the eigenvalues of the generalized eigenvalue problem*

$$\bar{A}x = \lambda \bar{W} x$$

lie in the closed interval $1 - \gamma^2 \leq \lambda \leq 1$, and $\sigma(I - \bar{W}^{-1}\bar{A}) \leq \gamma^2 < 1$.

Proof. The proof follows Theorem 6 of [1]. The theorem is an immediate consequence of the estimates

$$1 \leq \frac{x^T \bar{W} x}{x^T \bar{A} x} \leq \frac{1}{1 - \gamma^2}.$$

Since $\bar{W} = \bar{A} + \bar{L}\bar{D}^{-1}\bar{L}^T$ (note $\bar{U} = \bar{L}^T$) and $\bar{L}\bar{D}^{-1}\bar{L}^T$ is symmetric, positive semidefinite, it is clear that the lower bound is one. The upper bound is given by $1 + \mu$ where

$$\mu = \max_{x \neq 0} \frac{x^T \bar{L}\bar{D}^{-1}\bar{L}^T x}{x^T \bar{A} x}.$$

This can be written as

$$\mu = \max_{x \neq 0} \frac{y^T \bar{D} y}{x^T \bar{A} x},$$

where

$$\bar{D} y = \bar{L}^T x.$$

Consider the hierarchical decomposition of $\mathcal{V} = V_c \oplus V_f$. Let $\bar{a}(\cdot, \cdot)$ be symmetrized bilinear form corresponding to $\bar{A}$, and let $\|v\|^2 = \bar{a}(v, v)$. Then in finite element notation, we have

$$\mu = \max_{u \neq 0} \frac{\|\hat{v}\|^2}{\|u\|^2}, \tag{21}$$

where $u = v + w$, $v \in V_c$, $w \in V_f$ and $\hat{v} \in V_c$ satisfies

$$\bar{a}(\hat{v}, \chi) = \bar{a}(w, \chi) \tag{22}$$

for all $\chi \in V_c$. Written in finite element language, it is easy to analyse (21)-(22) in terms of the strengthened Cauchy inequality. We take $\chi = \hat{v}$ in (22) to see

$$\|\hat{v}\| \leq \gamma \|w\|.$$

On the other hand

$$\begin{aligned} \|u\|^2 &= \|v\|^2 + \|w\|^2 + 2\bar{a}(v, w) \\ &\geq \|v\|^2 + \|w\|^2 - 2\gamma \, \|v\| \, \|w\| \\ &\geq (1 - \gamma^2) \|w\|^2 \\ &\geq (1 - \gamma^2) \gamma^{-2} \|\hat{v}\|^2. \end{aligned}$$

The theorem now follows from combining this estimate and (21). □

7 Numerical results and conclusions

In this section we present numerical results for the interpolation coefficients analysed in the previous sections. We apply the schemes to the model convection-diffusion equation (1) with $f = 1$ on $\Omega = (0, 1) \times (0, 1)$. The problem is discretized using the Scharfetter-Gummel method. We perform experiments for the two-level method where the coarse grid is uniformly refined by dividing each triangle into four congruent triangles. We use uniform grids with structured refinement in order to treat all cases in a more standardized setting. We illustrate the dependence of the convergence rate on the direction and magnitude of βh.

We record average rates of convergence after $k = min\{100, \bar{k}\}$ iterations, where the residual was reduced by 10^{-4} in $\bar{k}$ steps. The average rate of convergence is given by $(\|r_k\|_2 / \|r_0\|_2)^{\frac{1}{k}}$, where r_i denotes the residual after i

steps. In the first table, we also record the improvement in the last step, i.e. $\|r_k\|_2/\|r_{k-1}\|_2$. These values are denoted in brackets. The entry 'zero' denotes the occurrence of a zero pivot in the underlying LU-decomposition. We choose $x_0 = (0, 0, \cdots, 0)^T$ as a starting vector. All calculations were done in double precision on a Sparc10.

$N = 3969$	Lin	G-S	S-G	ILU	hybrid
$(0, 0)$	0.19 (0.34)	0.99 (0.99)	0.19 (0.34)	0.99 (0.99)	0.19 (0.34)
$(0, 1000)$	0.52 (0.49)	0.74 (0.17)	0.70 (0.08)	0.35 (0.13)	0.35 (0.13)
$(0, 5000)$	0.52 (0.48)	0.69 (0.01)	0.70 (0.09)	0.09 (0.03)	0.09 (0.03)
$(707, 707)$	0.56 (0.56)	0.80 (0.41)	0.82 (0.10)	0.62 (0.17)	0.62 (0.17)
$(3536, 3536)$	0.56 (0.56)	0.80 (0.41)	zero (zero)	0.62 (0.17)	0.62 (0.17)
$(-707, 707)$	0.57 (0.63)	0.79 (0.41)	0.51 (0.28)	0.61 (0.17)	0.61 (0.17)
$(-3536, 3536)$	0.57 (0.63)	0.79 (0.41)	0.51 (0.28)	0.61 (0.17)	0.61 (0.17)

Table 2. Convergence rates for various two-level methods and for several values of $\beta = (\beta_1, \beta_2)^T$. ("lin" is linear interpolation, "G-S" is Gauß-Seidel, "S-G" is exponential interpolation and "ILU" is ILU interpolation.

$\beta = (0, 1000)^T$	Lin	G-S	S-G	ILU	hybrid
$h = 1/8$ ($N = 49$)	0.45	0.02	0.44	e-3	e-3
$h = 1/16$ ($N = 225$)	0.50	0.22	0.45	0.01	0.01
$h = 1/32$ ($N = 961$)	0.51	0.56	0.49	0.08	0.08
$h = 1/64$ ($N = 3969$)	0.52	0.74	0.70	0.35	0.35
$h = 1/128$ ($N = 16129$)	0.53	0.85	0.86	0.63	0.63
$\beta = (707, 707)^T$	Lin	G-S	S-G	ILU	hybrid
$h = 1/8$ ($N = 49$)	0.50	e-3	zero	e-5	e-5
$h = 1/16$ ($N = 225$)	0.56	0.39	zero	0.19	0.19
$h = 1/32$ ($N = 961$)	0.56	0.64	0.66	0.36	0.36
$h = 1/64$ ($N = 3969$)	0.56	0.80	0.82	0.62	0.62
$h = 1/128$ ($N = 16129$)	0.55	0.89	0.90	0.79	0.79

Table 3. Convergence rates for various two-level methods and for several values of h with fixed $\beta = (0, 1000)^T$ (top) and $\beta = (707, 707)^T$ (bottom).

Typical convergence histories are shown in Figure 4, where $\log \|r_i\|/\|r_0\|$ is plotted as a function of the iteration index i. Here we observe the non-monotonic behavior of the residual for the Scharfetter-Gummel factors.
The results for the two level method let the linear factors appear very favorable. We point out that this is only the case for two levels. For more than two levels, the method is applied recursively for the calculated coarse grid approximation. Unfortunately, the coarse grid matrix usually does not correspond to a discretization of our model problem (1) on the coarse grid anymore. For linear factors as well as for the Scharfetter-Gummel factors, we

obtain seven point formulae describing possibly indefinite systems. For these the above considerations about including the geometry of the mesh or the coefficients of the problem into the calculation of the interpolation factors do not apply. Thus it is not surprising that our numerical experiments for these methods for more than two levels usually failed.
On the other side, the ILU-factors seem to provide a robust scheme for more than two levels. The resulting coarse grid matrix will be diagonally dominant, and the factors for further unrefinement are calculated using the entries of this coarse grid matrix. In our numerical experiments we observed that the convergence rates hardly decreased when switching from two to several levels.

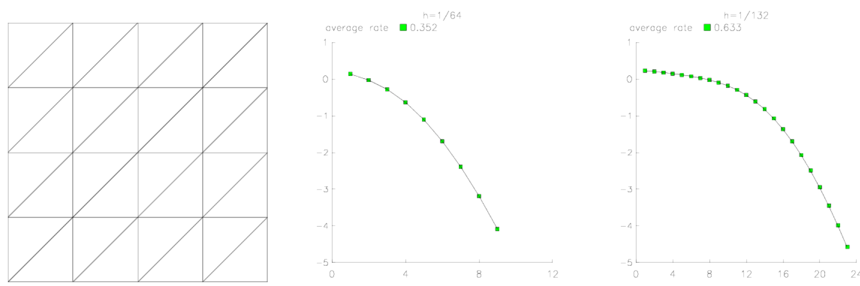

Fig. 4. The initial mesh (1^{st}) and convergence histories for ILU, $\beta = (0, 1000)^T$, where $h = 1/64$ (2^{nd}) and $h = 1/128$ (3^{rd})

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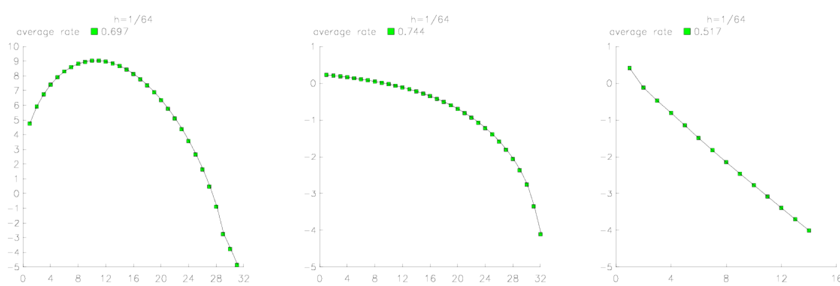

Fig. 5. Convergence histories with $\beta = (0, 1000)^T$ for Scharfetter-Gummel where $h = 1/64$, (1^{st}), Gauß-Seidel, $h = 1/64$ (2^{nd}) and Linear, $h = 1/64$ (3^{rd}).

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